Topic 1: The Solow Model of Economic Growth

Macroeconomics is not a one-size-fits-all type of field. It would be a duanting task to even attempt to construct a model that explained all interesting macroeconomic phenomena, and any such model would undoubtedly be complicated and unwieldy, making it difficult to learn (and teach). For this reason, macroeconomists tend to adopt a more eclectic approach, with models often being developed with the intention of helping to explain one particular aspect of macroeconomy.

The first model that we will look at in this class, a model of economic growth originally developed by MIT's Robert Solow in the 1950s, is a good example of this general approach. Solow's purpose in developing the model was to deliberately ignore some important aspects of macroeconomics, such as short-run fluctuations in employment and savings rates, in order to develop a model that attempted to describe the long-run evolution of the economy. The resulting paper (A Contribution to the Theory of Economic Growth, *QJE*, 1956) remains highly influential even today and, despite its relative simplicity, it conveys a number of very useful insights about the dynamics of the growth process.

The Solow model is also worth teaching from a methodological perspective because it provides a simple example of the type of dynamic model that is commonly used in today's more advanced macroeconomic theory: For those brave souls amongst you who may be interested in studying more macroeconomics after this course, note that some of the tricks and terminology introduced here (such as the use logs to derive growth rates, and the concepts of steady-state growth and convergent dynamics) are widespread in modern macroeconomics.

An Aside on Notation

We are interested in modelling changes over time in outputs and inputs. A useful mathematical shorthand that saves us from having to write down derivatives with respect to time everywhere is to write

$$\dot{Y}_t = \frac{dY_t}{dt} \tag{1}$$

What we are really interested in, though, is *growth rates* of series: If I tell you GDP was up by 5 million euros, that may sound like a lot, but unless we scale it by the overall level of GDP, it's not really very useful information. Thus, what we are interested in calculating is $\frac{\dot{Y}_t}{Y_t}$, and this is our mathematical expression for the growth rate of a series.

The Solow Model's Ingredients

The model assumes that GDP is produced according to an aggregate production function technology. It is worth flagging that most of the key results for Solow's model can be obtained using any of the standard production functions that you see in microeconomic production theory. However, for concreteness, I am going to be specific and limit us to the case in which the production function takes the Cobb-Douglas form:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} \qquad 0 < \alpha < 1 \tag{2}$$

where K_t is capital input and L_t is labour input. Note that an increase in A_t results in higher output without having to raise inputs. Macroeconomists tend to call increases in A_t "technological progress" and we will loosely refer to this as the "technology" term, but ultimately A_t is simply a measure of productive efficiency. Because an increase in A_t increases the productiveness of the other factors, it is also sometimes known as *Total Factor Productivity* (TFP), and this is the term most commonly used in empirical papers that attempt to calculate this series.

In addition to the production function, the model has four other equations.

- Capital accumulates according to

$$\dot{K}_t = Y_t - C_t - \delta K_t \tag{3}$$

  In other words, the addition to the capital stock each period depends positively on savings (this is a closed-economy model so savings equals investment) and negatively on depreciation, which is assumed to take place at rate δ.

- Labour input grows at rate n:

$$\frac{\dot{L}_t}{L_t} = n \tag{4}$$

- Technological progress grows at rate g:

$$\frac{\dot{A}_t}{A_t} = g$$

- A fraction s of output is saved each period.

$$Y_t - C_t = sY_t \tag{5}$$

We have not put time subscripts on the rate of population growth, the rate of technological progress, the rate of depreciation of capital or the savings rate, because we will generally consider these to be constant: The Solow model does not attempt to explain fluctuations in these variables. However, we do wish to charaterise the dynamics of the model well enough to be able to figure out what happens if these parameters changes. So, for instance, we will be interested in what happens when there is a once-off increase in the savings rate.

A Digression on the Production Function

Two well-known features of the Cobb-Douglas production function are worth recapping here:

- Constant returns to scale (a doubling of inputs leads to a doubling of outputs):

$$A_t(\mu K_t)^{\alpha}(\mu L_t)^{1-\alpha} = \mu^{\alpha}\mu^{1-\alpha}Y_t = \mu Y_t \tag{6}$$

- Decreasing marginal returns to factor accumulation (adding extra capital while holding labour input fixed yields ever-smaller increases in output):

$$\frac{\partial Y}{\partial K} = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha} \tag{7}$$

$$\left(\frac{\partial^2 Y}{\partial K}\right) = \alpha\left(\alpha - 1\right) A_t K_t^{\alpha-2} L_t^{1-\alpha} < 0 \tag{8}$$

  This turns out to be the key element of the model. Think about why it is sensible: If a firm acquires an extra unit of capital, it should raise its output. But if the firm keeps piling on extra capital without raising the number of workers available to use this capital, the increases in output will probably taper off. In the Cobb-Douglas case, the parameter α dictates the pace of this tapering off.

Modelling Dynamics using Tricks with Logs

One of the purposes of this class is to teach you some tricks that macroeconomists use when constructing and solving models. So here's the first: The use of logarithims can be very helpful when dealing with dynamic models. One reason for this is the following property:

$$\frac{d\left(\log Y_t\right)}{dt} = \frac{d\left(\log Y_t\right)}{dY_t}\frac{dY_t}{dt} = \frac{\dot{Y}_t}{Y_t} \tag{9}$$

The growth rate of a series is the same as the derivative of its log with respect to time (note the use of chain-rule of differentiation in the above equation.)

Two other useful properties of logarithms that will also help us characterise the dynamics of the Solow model are the following:

$$\log(XY) = \log X + \log Y \tag{10}$$

$$\log\left(X^Y\right) = Y \log X \tag{11}$$

We can apply each of these properties to get a useful representation of the growth rate of output:

$$\log(Y_t) = \log(A_t K_t^{\alpha} L_t^{1-\alpha}) \tag{12}$$

$$= \log(A_t) + \log(K_t^{\alpha}) + \log(L_t^{1-\alpha}) \tag{13}$$

$$= \log(A_t) + \alpha \log(K_t) + (1-\alpha)\log(L_t) \tag{14}$$

Now taking the derivative with respect to time, we get the required formula:

$$\frac{\dot{Y}_t}{Y_t} = \frac{\dot{A}_t}{A_t} + \alpha \frac{\dot{K}_t}{K_t} + (1-\alpha)\frac{\dot{L}_t}{L_t} \tag{15}$$

This takes us from the Cobb-Douglas formula involving levels to a simple formula involving growth rates. The growth rate of output per worker is simply

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{A}_t}{A_t} + \alpha\left(\frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t}\right) \tag{16}$$

Thus, there are two sources of increases in output per worker:

- Technological progress
- Capital deepening (i.e. increases in capital per worker)

The Solow model provides a useful framework for understanding how technological progress and capital deepening interact to determine the growth rate of output per worker.

Steady-State Growth

The first thing we are going to do with the Solow model is figure out what this economy looks like along a path on which output growth is constant. Macroeconomists refer to such constant growth paths as *steady-state* growth paths. We don't necessarily want to study *only* constant-growth paths, but we will see below that the Solow-model economy tends to converge over time towards this path.

First note that, given constant growth rates for technology and labour input, all variations in output growth are due to variations in the growth rate of capital input:

$$\frac{\dot{Y}_t}{Y_t} = g + \alpha\frac{\dot{K}_t}{K_t} + (1-\alpha)n \tag{17}$$

So for output growth to be constant, we must also have capital growth being constant.

We can also show that these growth rates for capital and output must be the same, so that the capital-output ratio is constant along a constant growth. To see this, re-write the capital accumulation equation as

$$\dot{K}_t = sY_t - \delta K_t \tag{18}$$

and divide across by K_t on both sides

$$\frac{\dot{K}_t}{K_t} = s\frac{Y_t}{K_t} - \delta \tag{19}$$

The growth rate of the capital stock depends negatively on the capital-output ratio $\frac{K_t}{Y_t}$. So, for the capital stock to be growing at a constant rate, then $\frac{K_t}{Y_t}$ must be constant. But $\frac{K_t}{Y_t}$ can only be constant if the growth rate of K_t is the same as the growth rate of Y_t.

With this result in mind, we see that the steady-state growth rate must satisfy

$$\frac{\dot{Y}_t}{Y_t} = g + \alpha\frac{\dot{Y}_t}{Y_t} + (1-\alpha)n \tag{20}$$

Subtracting $\alpha\frac{\dot{Y}_t}{Y_t}$ from both sides, we get

$$(1-\alpha)\frac{\dot{Y}_t}{Y_t} = g + (1-\alpha)n \tag{21}$$

So, the steady-state growth rate is

$$\frac{\dot{Y}_t}{Y_t} = \frac{g}{1-\alpha} + n \tag{22}$$

Only the growth rate of technology, g, and the factor controlling the extent of diminishing marginal returns to capital, α, can affect the growth rate of output per worker.

This is a key result: All the other parameters have *no effect* on the steady-state growth rate. For example, economies with higher saving rates do not have faster steady-state growth rates. Why is this? An increase in the saving rate can raise the growth rate initially by boosting capital accumulation. But diminishing marginal returns implies that during this period capital growth will outstrip output growth. And this will not last: Equation (19) tells us that capital growth depends negatively on the capital-output ratio. So higher saving rates can produce temporary increases the growth rate of output, but canot get the economy to a path involving a faster *steady-state* growth rate.

Growth Accounting versus Growth Theory

This result also illustrates an important qualification to the decomposition of growth into capital deepening and technological progress given by equation (16). Such decompositions (known as *growth accounting* studies) are quite commonly carried out, with researchers concluding that a certain fraction of the growth in output per worker over a certain period was due to capital deepening. However, these calculations are based on an arithmetic decomposition of the growth in output per worker, and should not be confused with an economic theory explaining the ultimate sources of growth.

In particular, the Solow model tells us that we should be careful not to draw from such calculations that policies based *solely* on encouraging capital deepening are capable of boosting the growth rate in the long run. Diminishing marginal productivity of capital implies that steady growth can not be maintained based on capital deepening alone. Ultimately, it is technological progress that offsets the effects of diminishing marginal returns, and thus allows capital deepening to play a role along the steady growth path.

A Useful Expression for Output Per Worker

Our next goal is to provide a full characterisation of the dynamics of capital and output in the Solow model. To do this, there is a particular characterisation of output per worker that turns out to be very useful. First, let us define the capital-output ratio as

$$x_t = \frac{K_t}{Y_t} \tag{23}$$

So, the production function can be expressed as

$$Y_t = A_t \left(x_t Y_t\right)^{\alpha} L_t^{1-\alpha} \tag{24}$$

Here, we are using the fact that

$$K_t = x_t Y_t \tag{25}$$

Dividing both sides of this expression by Y_t^α, we get

$$Y_t^{1-\alpha} = A_t x_t^\alpha L_t^{1-\alpha} \tag{26}$$

Taking both sides of the equation to the power of $\frac{1}{1-\alpha}$ we arrive at

$$Y_t = A_t^{\frac{1}{1-\alpha}} x_t^{\frac{\alpha}{1-\alpha}} L_t \tag{27}$$

So, output per worker is

$$\frac{Y_t}{L_t} = A_t^{\frac{1}{1-\alpha}} x_t^{\frac{\alpha}{1-\alpha}} \tag{28}$$

This equation tells us that all fluctuations in output per worker are due to either changes in technological progress or changes in the capital-output ratio. Because A_t is assumed to grow at a constant rate each period, this means that all of the interesting dynamics for output per hour stem from the behaviour of the capital-output ratio. We will now describe how this ratio behaves.

Dynamics of the Capital-Output Ratio

Using our terminology for the capital-output ratio, we can re-write equation (19) as

$$\frac{\dot{K}_t}{K_t} = \frac{s}{x_t} - \delta \tag{29}$$

Again using logarithm tricks, note that

$$\log x_t = \log \frac{K_t}{Y_t} = \log K_t + \log \frac{1}{Y_t} = \log K_t + \log Y_t^{-1} = \log K_t - \log Y_t \tag{30}$$

Taking derivatives with respect to time we have

$$\frac{\dot{x}_t}{x_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{Y}_t}{Y_t} \tag{31}$$

Now using equation (17) for output growth and equation (29) for capital growth, we can derive a useful equation for the dynamics of the capital-output ratio:

$$\frac{\dot{x}_t}{x_t} = (1-\alpha)\frac{\dot{K}_t}{K_t} - g - (1-\alpha)n \tag{32}$$

$$= (1-\alpha)\left(\frac{s}{x_t} - \frac{g}{1-\alpha} - n - \delta\right) \tag{33}$$

This dynamic equation has a very important property: The growth rate of x_t depends negatively on the value of x_t. In particular, when x_t is over a certain value, it will tend to decline, and when it is under that value it will tend to increase. Thus the capital-ouput ratio exhibits *convergent dynamics*: It tends to converge to a specific long-run steady-state value.

What is this long-run value, which we will label x^*? It is the value consistent with $\frac{\dot{x}}{x} = 0$. This implies that

$$\frac{s}{x^*} - \frac{g}{1-\alpha} - n - \delta = 0 \tag{34}$$

This solves to give

$$x^* = \frac{s}{\frac{g}{1-\alpha} + n + \delta} \tag{35}$$

Given this expression for the steady-state capital-output ratio, we can also derive a more intuitive-looking expression to describe the convergence properties of the ratio. We do this by both multiplying and dividing the right-hand-side of equation (33) by $(\frac{g}{1-\alpha} + n + \delta)$:

$$\frac{\dot{x}_t}{x_t} = (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{s/x_t - \frac{g}{1-\alpha} - n - \delta}{\frac{g}{1-\alpha} + n + \delta}\right) \tag{36}$$

The last term inside the brackets can be simplified to give

$$\begin{aligned}
\frac{\dot{x}_t}{x_t} &= (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{1}{x_t}\frac{s}{\frac{g}{1-\alpha} + n + \delta} - 1\right) && (37)\\
&= (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{x^*}{x_t} - 1\right) && (38)\\
&= (1-\alpha)(\frac{g}{1-\alpha} + n + \delta)\left(\frac{x^* - x_t}{x_t}\right) && (39)
\end{aligned}$$

This equation states that each period the capital-output ratio closes a fraction equal to $(1-\alpha)(\frac{g}{1-\alpha} + n + \delta)$ of the gap between the current value of the ratio and its steady-state value.

The Steady-State Level of Output Per Worker

We have derived the dynamic behaviour of the capital-output ratio in the Solow model. It turns out to be pretty easy to also derive the model's predictions for the behaviour of output per worker. This is because output per worker is determined by A_t, which we know follows a set path, and by the capital-output ratio, whose dynamics we have just derived.

To see this, apply the take-logs-and-derivatives trick to equation (28) to get the growth rate of output per worker in terms of technological progress and changes in the capital-output ratio:

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{1}{1-\alpha}\frac{\dot{A}_t}{A_t} + \frac{\alpha}{1-\alpha}\frac{\dot{x}_t}{x_t} \tag{40}$$

Substituting in the growth rate of x from equation (39) to get

$$\frac{\dot{Y}_t}{Y_t} - \frac{\dot{L}_t}{L_t} = \frac{g}{1-\alpha} + \alpha(\frac{g}{1-\alpha} + n + \delta)\left(\frac{x^* - x_t}{x_t}\right) \tag{41}$$

Output growth equals the steady-state growth rate $\frac{g}{1-\alpha}$ plus or minus that element due to the capital-output ratio converging towards its steady-state level.

The model also gives us an expression for the steady-state path for output per worker, i.e. the path towards which output is always converging in which $x_t = x^*$. This is obtained by plugging the steady-state capital-output ratio into equation (28) to get

$$\left(\frac{Y_t}{L_t}\right)^* = A_t^{\frac{1}{1-\alpha}}\left(\frac{s}{\frac{g}{1-\alpha} + n + \delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{42}$$

This formula provides a way to calculate the long-run effects on the level of output per worker of changes in the savings rate, depreciation rate etc.

Concrete Example 1: Convergence Dynamics

Often, the best way to understand dynamic models is to load them onto the computer and see them run. This is easily done using spreadsheet software such as Excel or econometrics-oriented packages such as RATS. Figures 1 to 3 provide examples of the behaviour over time of two economies, one that starts with a capital-output ratio that is half the steady-state level, and other that starts with a capital output ratio that is 1.5 times the steady-state level.

The parameters chosen were $s = 0.2, \alpha = \frac{1}{3}, g = 0.02, n = 0.01, \delta = 0.06$. Together these parameters are consistent with a steady-state capital-output ratio of 2. To see, this plug these values into (35):

$$\left(\frac{K}{Y}\right)^* = \frac{s}{\frac{g}{1-\alpha} + n + \delta} = \frac{0.2}{1.5 * 0.02 + 0.01 + 0.06} = 2 \tag{43}$$

The first chart shows how the two capital-output ratios converge, somewhat slowly, over time to their steady-state level. This slow convergence is dictated by our choice of parameters: Our "convergence speed" is:

$$\lambda = (1-\alpha)(\frac{g}{1-\alpha} + n + \delta) = \frac{2}{3}(1.5 * 0.02 + 0.01 + 0.06) = 0.067 \tag{44}$$

So, the capital-output ratio converges to its steady-state level at a rate of about 7 percent per period. These are fairly standard parameter values for annual data, so this should be understood to mean 7 percent per year.

The second chart shows how output per worker evolves over time in these two economies. Both economies exhibit growth, but the capital-poor economy grows faster during the convergence period than the capital-rich economy. These output per worker differentials may seem a little small on this chart, but the final chart shows the behaviour of the growth rates, and this chart makes it clear that the convergence dynamics can produce substantially different growth rates depending on whether an economy is above or below its steady-state capital-output ratio. During the initial transition periods, the capital-poor economy grows at rates over 6 percent, while the capital-rich economy grows at under 2 percent. Over time, both economies converge towards the steady-state growth rate of 3 percent.

Concrete Example 2: Changes in Parameters

Figures 4 to 6 examine what happens when the economy is moving along the steady-state path consistent with the parameters just given, and then one of the parameters is changed. Specifically, it examines the effects of changes in s, δ and g.

Consider first an increase in the savings rate to $s = 0.25$. This has no effect on the steady-state growth rate. But it does change the steady-state capital-output ratio from 2 to 2.5. So the economy now finds itself with too little capital relative to its new steady-state capital-output ratio. The growth rate jumps immediately and only slowly returns to the long-run 3 percent value. The faster pace of investment during this period gradually brings the capital-output ratio into line with its new steady-state level.

The increase in the savings rate permamently raises the level of output per worker relative to the path that would have occurred without the change. However, for our parameter values, this effect is not that big. This is because the long-run effect of the savings rate on output per worker is determined by $s^{\frac{\alpha}{1-\alpha}}$, which in this case is $s^{0.5}$. So in our case, 25 percent increase in the savings rate produces an 11.8 percent increase in output per worker ($1.25^{0.5} = 1.118$). More generally, a doubling of the savings rate raises output per worker by 41 percent ($2^{0.5} = 1.41$).

The charts also show the effect of an increase in the depreciation rate to $\delta = 0.11$. This reduces the steady-state capital-output ratio to 4/3 and the effects of this change are basically the opposite of the effects of the increase in the savings rate.

Finally, there is the increase in the rate of technological progress. I've shown the effects of a change from $g = 0.02$ to $g = 0.03$. This increases the steady-state growth rate of output per worker to 0.045. However, as the charts show there is another effect: A faster steady-state growth rate for output reduces the steady-state capital-output ratio. Why? The increase in g raises the long-run growth rate of output; this means that each period the economy needs to accumulate more capital than before just to keep the capital-output ratio constant. Again, without a change in the savings rate that causes this to happen, the capital-output ratio will decline. So, the increase in g means that—as in the depreciation rate example—the economy starts out in period 25 with too much capital relative to its new steady-state capital-output ratio. For this reason, the economy doesn't jump straight to its new 4.5 percent growth rate of output per hour. Instead, after an initial jump in the growth rate, there is a very gradual transition the rest of the way to the 4.5 percent growth rate.

Figure 1

Convergence Dynamics for the Capital-Output Ratio

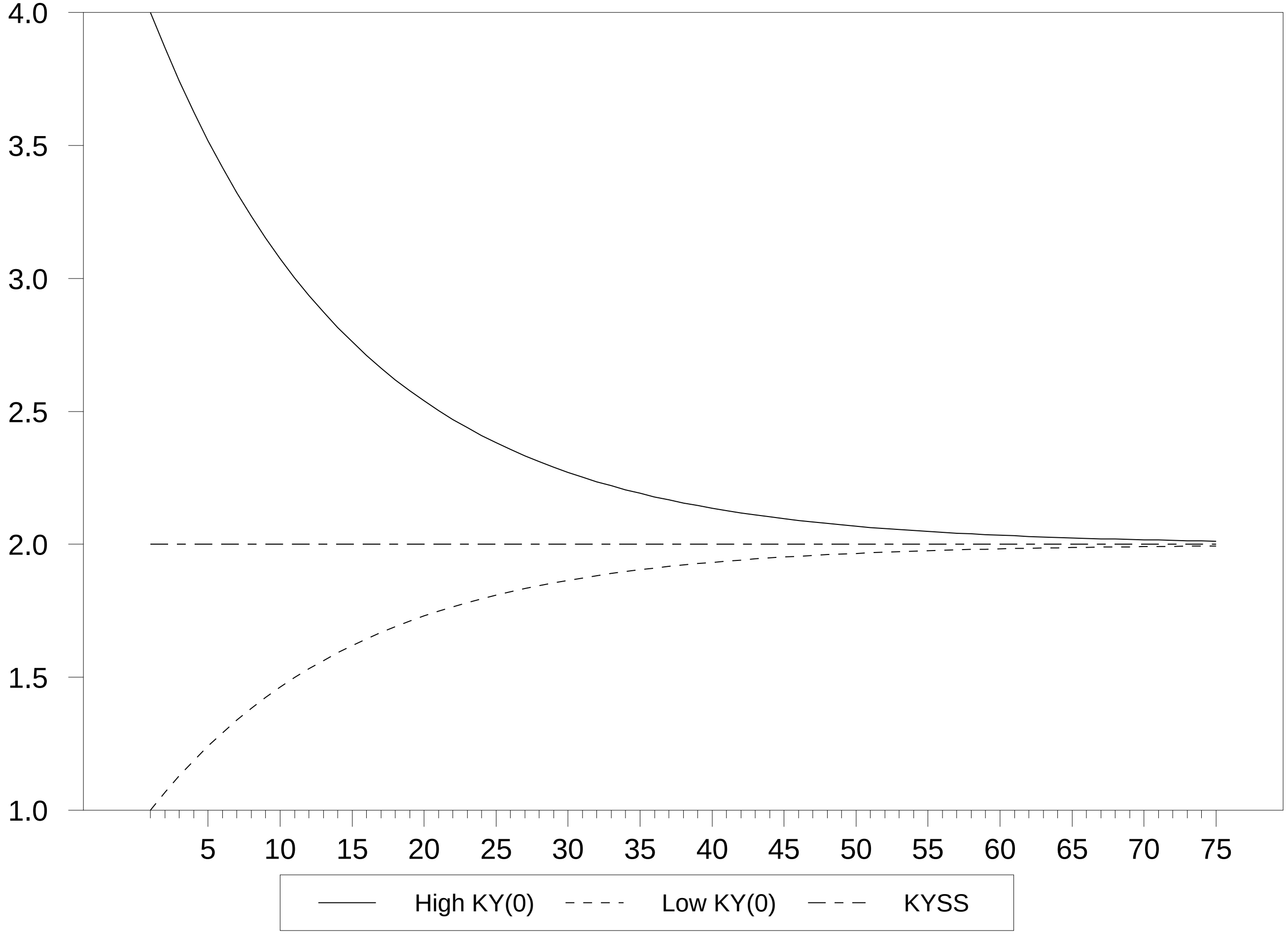

Figure 2

Convergence Dynamics for Output Per Worker

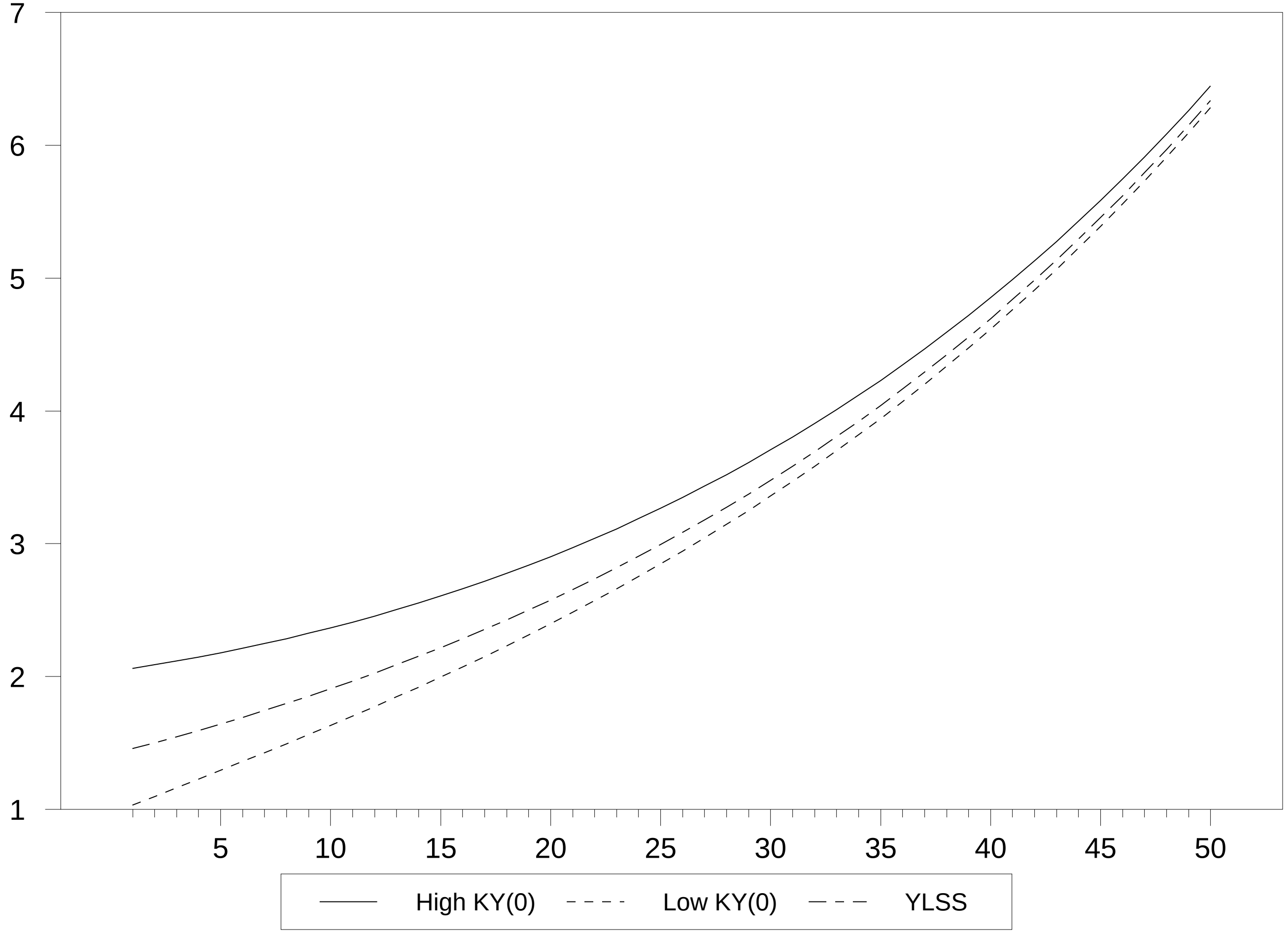

Figure 3

Convergence Dynamics for Growth Rates of Output Per Worker

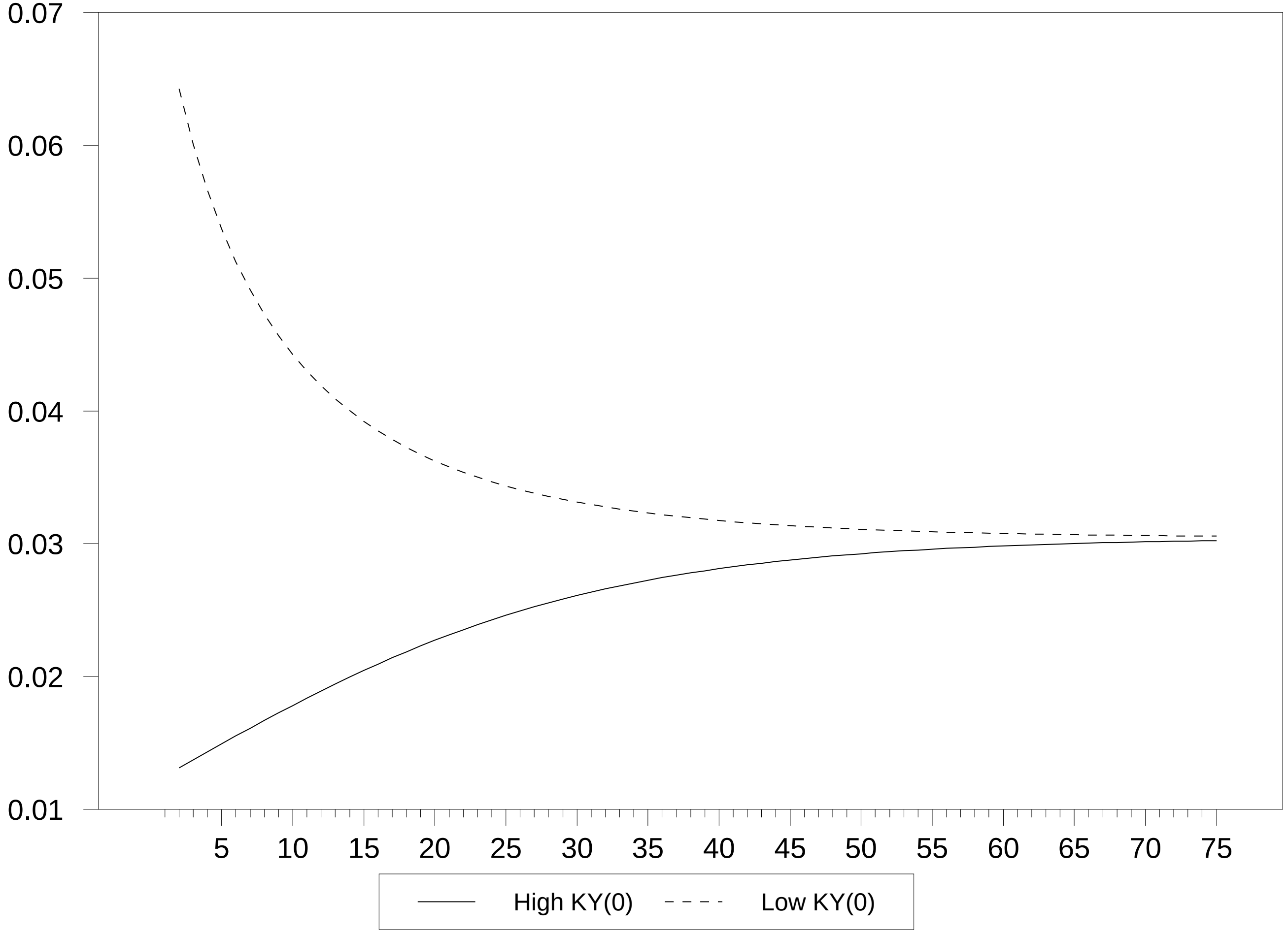

Figure 4

Capital-Output Ratios: Effects of Increases in ....

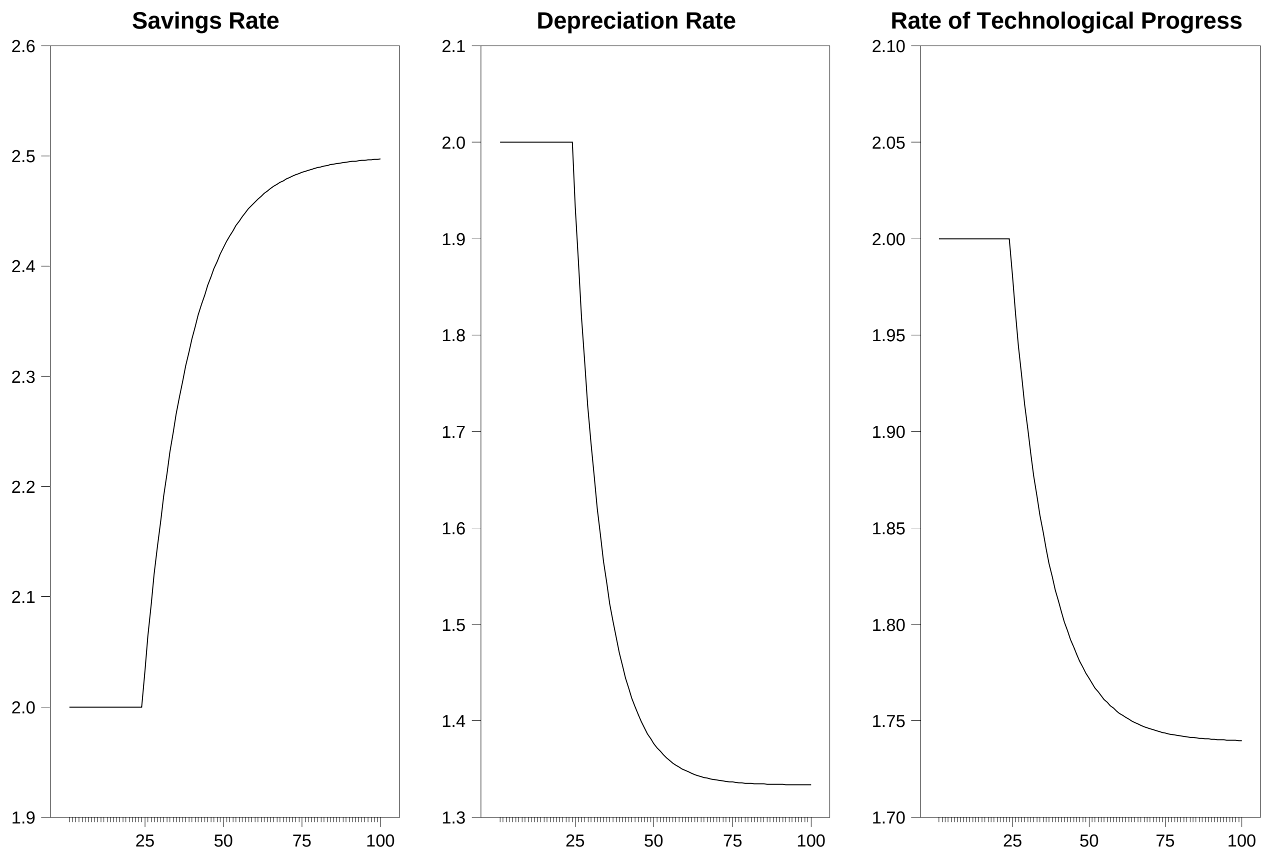

Figure 5

Growth Rates of Output Per Hour: Effects of Increases in ....

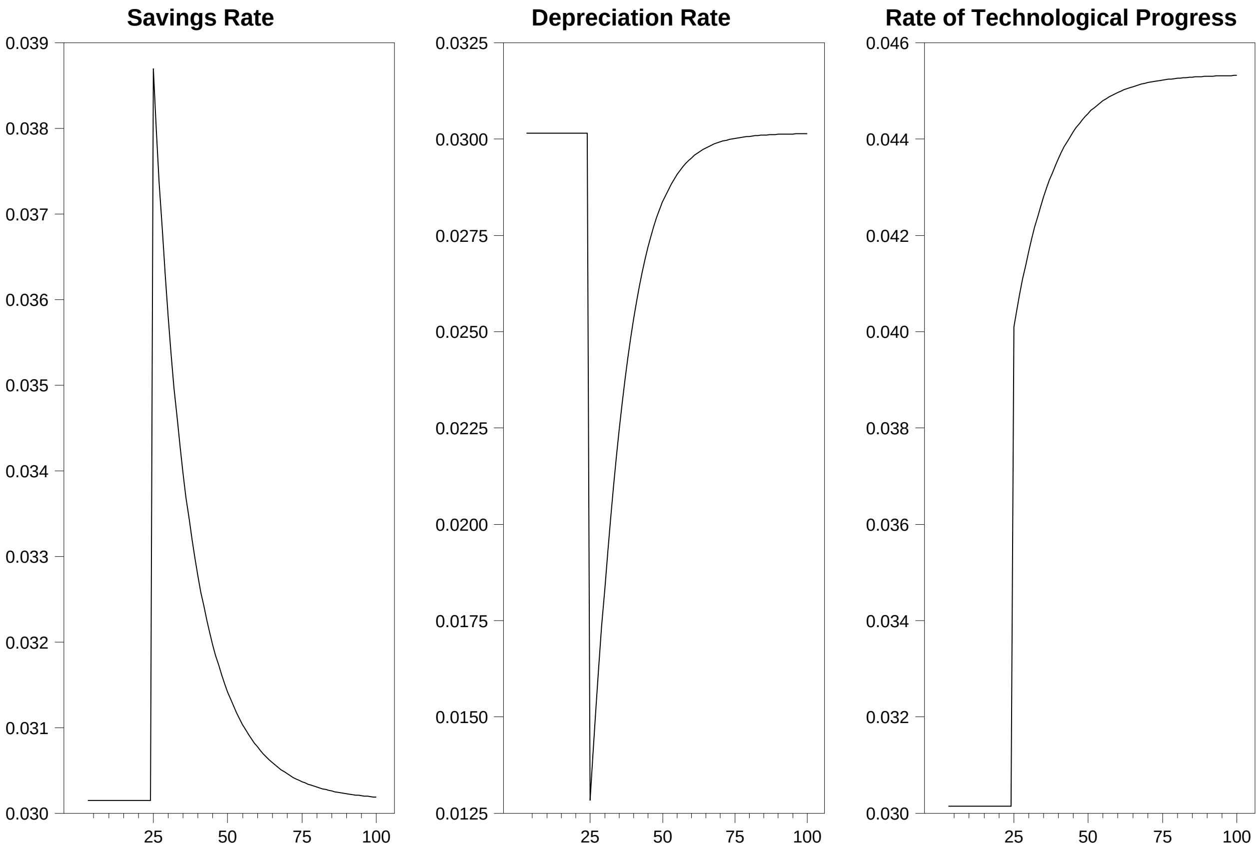

Figure 6

Output Per Hour: Effects of Increases in ....