

Spillovers and Strategic Cooperative Behaviour

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Abstract

In this paper, we analyse cooperative situations in which the formation of a coalition results in spillovers to the players outside that coalition. To this type of situation a new type of game, a *spillover game*, is associated, in which both cooperative and strategic elements play a role. Some basic game theoretical concepts are analysed for this new class of games and illustrated by means of three applications of our model: government situations, public-private connection problems and cartels in oligopolistic markets.

1 Introduction

In standard noncooperative game theory it is assumed that players cannot make binding agreements. That is, each cooperative outcome must be sustained by Nash equilibrium strategies. At the other end of the spectrum, in cooperative game theory, players have no choice but to cooperate. The standard transferable utility (TU) model assumes that all players involved want to come to an agreement and the main task is to propose socially acceptable solutions. Noncooperative theory tries to predict the outcome of strategic situations using equilibrium concepts that at least require the predicted strategy combinations to be robust against unilateral deviations.

Both approaches seem to be diametrically opposed. Many real life situations, however, exhibit both cooperative and strategic features. Neither approach suffices in these cases. Examples of these situations can be found in parliaments where governments are based on multiple-party coalitions. Here noncooperative theory obviously does not work, since agreements have to be made. Also TU theory is not sufficiently rich, since typically not all parties represented in parliament are part of the government. Furthermore, TU theory does not take into account the spillover effects from coalitions on the parties outside. These spillovers measure the impact

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of government policy on the opposition parties and thus reflect in some way the parties' relative positions in the political spectrum.

This kind of spillovers is present in many situations. For example, one can think of a situation where a group of people needs to be connected to a source, like in a telecommunication network. In the literature, many solution concepts have been introduced (cf. Bird (1976)), but these do not take into account the strategic considerations of players not to join in the public enterprise. However, spillovers occur if one assumes that publicly accessible networks can be built by smaller groups, hence creating a special type of free-rider effect. In order to find a fair solution, these possible spillovers should be taken into account.

As another example, one can think of a group of firms that compete in an oligopolistic industry. In assessing the benefits of a cartel, a crucial part is played by comparing the profit when joining the cartel and when staying out. Our spillover model allows for the analysis of all intermediate cases in between a completely non-cooperative (eg, Cournot-Nash) situation and a fully cooperative (cartel) situation.

From the aforementioned examples one can conclude that in many cooperative situations, a socially acceptable solution concept should incorporate the strategic options that result from spillovers. Essentially, spillovers induce a noncooperative aspect in cooperative situations. They provide incentives for players to join or to stay out of a coalition. In TU games, these spillovers are not taken into account, but one implicitly assumes that players do not have a better alternative than to stay in the group. As mentioned before, in the government example, this is typically not the case. In short, cooperative game theory lacks the strategic outside options players have, whereas noncooperative theory, on the other hand, does not allow for explicit cooperation.

To capture spillovers in a cooperative model, a new class of games is introduced, namely *spillover games*. This class of games builds on ideas introduced in Van der Rijt (2000) for government situations. In a spillover game, each coalition is assigned a value, as in a TU game. In addition, all the players outside the coalition are separately assigned a value as well, capturing the spillovers from the coalition to the outside players. We restrict ourselves to a coalitional structure where there is one coalition (eg, a government, a group building a public network, a cartel) and a group of singletons outside. This allows us to redefine some basic concepts of TU theory, while not assuming *ex ante* that all players are fully cooperating.

The model of spillover games is explicitly aimed at analysing the influence of a coalition S on the payoffs of the players outside S . In this sense, spillover games differ fundamentally from games in partition function form (cf. Bloch (1996) and Yi

(1997)), where for each coalition S the influence of the possible coalition structures on the player set outside S on the payoff to coalition S is analysed. Hence, the causality of spillovers in spillover games is reversed compared to partition function form games.

The structure of the paper is as follows. In Section 2 the class of spillover games is introduced. Our three main applications are presented: government situations, public-private connection problems, and cartels in oligopolistic markets. In Section 3 we introduce the core concept for the class of spillover games and present a balancedness result reminiscent of TU theory. In Section 4 government situations are analysed in more detail. A power index is introduced that is constructed in a similar way as the Shapley value for TU games. Section 5 takes a closer look at public-private connection problems and introduces a notion of convexity for spillover games. Finally, Section 6 deals with the Cournot model for oligopolistic markets and shows how this type of noncooperative games can be transformed into a spillover game in a natural way.

2 The model

A *spillover game* is a tuple $\mathcal{G} = (N, \mathcal{W}, v, z)$, where $N = \{1, \dots, n\}$ is the set of players, $\mathcal{W} \subset 2^N$ is a set of coalitions that can cooperate and v and z are payoff functions, to be specified below.

One main feature of our model is the assumption that exactly one coalition of players will cooperate. Contrary to TU games, however, we do not impose that the resulting coalition is the grand coalition. In the example of government formation, the grand coalition would be a very extreme outcome.

The set $\mathcal{W} \subset 2^N$ contains those coalitions which can actually cooperate. An element of \mathcal{W} is called a *winning coalition*. In a government situation, a natural choice for \mathcal{W} is the collection of coalitions which have a majority in parliament.

We assume that \mathcal{W} satisfies the following properties:

- $N \in \mathcal{W}$.
- $S \subset T, S \in \mathcal{W} \Rightarrow T \in \mathcal{W}$ (monotonicity).

The first property ensures that the game is not trivial, in the sense that there is at least one winning coalition. The second property states that if a small group of players S can cooperate (eg, have a majority), then a larger coalition $T \supset S$ is also winning.

The (nonnegative) payoff function $v : 2^N \rightarrow \mathbb{R}_+$ assigns to every coalition $S \subset N$ a value $v(S)$. If $S \in \mathcal{W}$, then $v(S)$ represents the total payoff to the members of S in case they cooperate. For $S \notin \mathcal{W}$ we simply impose $v(S) = 0$.

Suppose that the players in S cooperate. Then the members of S do not only generate a payoff to themselves. Their cooperation also affects the players outside S . The payoffs to the other players, which are called *spillovers* (wrt S), are given by the vector $z^S \in \mathbb{R}_+^{N \setminus S}$. Again, we simply put $z^S = 0$ for $S \notin \mathcal{W}$. Note that whereas the members of S still have the freedom to divide the amount $v(S)$ among themselves, the payoffs to the players outside S are individually fixed.

Spillovers (wrt S) are called *positive*, if the total payoff to every coalition $U \subset N \setminus S$ is higher than what U can earn on its own, so if

$$\sum_{i \in U} z_i^S \geq v(U) \quad (1)$$

for every $U \subset N \setminus S$. Likewise, spillovers are *negative* if for every $U \subset N \setminus S$ the reverse inequality holds in (1). Note that if for different coalitions U not the same inequality holds, spillovers are neither positive nor negative.

A set of winning coalitions $\mathcal{W} \subset 2^N$ is called *N-proper* if $S \in \mathcal{W}$ implies $N \setminus S \notin \mathcal{W}$. In the context of coalition formation in politics, this property relates to the fact that a coalition and its complement can not have a majority at the same time.

In the remainder of this section, we provide three examples to illustrate the spillover model.

Example 1 Consider a parliament with four parties¹: the communists (COM), socialists (SOC), Christian democrats (CD) and liberals (LIB). The seats are divided as follows:

party	COM	SOC	CD	LIB
share of seats	0.1	0.3	0.25	0.35

This gives rise to a spillover game with $N = \{COM, SOC, CD, LIB\}$ and an *N-proper* set \mathcal{W} of coalitions having a majority:

$$\mathcal{W} = \{\{SOC, CD\}, \{SOC, LIB\}, \{CD, LIB\}, \{COM, SOC, CD\}, \\ \{COM, SOC, LIB\}, \{COM, CD, LIB\}, \{SOC, CD, LIB\}, N\}.$$

¹This example is inspired by the model presented in Van der Rijt (2000).

For the winning coalitions the payoffs could look as follows (the first entry in the two-dimensional z -vectors corresponds to COM):²

S	$v(S)$	z^S
$\{SOC, CD\}$	12	(4, 3)
$\{SOC, LIB\}$	10	(2, 7)
$\{CD, LIB\}$	15	(0, 4)
$\{COM, SOC, CD\}$	19	0
$\{COM, SOC, LIB\}$	13	6
$\{COM, CD, LIB\}$	14	4
$\{SOC, CD, LIB\}$	18	1
N	16	

Obviously, COM and LIB do not have much in common, which is reflected by a relatively low payoff to coalitions in which both are involved. The central position of CD is reflected by the relatively high spillover it experiences when a coalition forms in which it is not involved. If all four parties get together, the result will not be stable, which is reflected by the low value for N .

Example 2 Consider a group of players that can be connected to a source. If a player is connected to the source, he receives some fixed benefit. On the other hand, by creating connections costs are incurred. Each player can construct a direct link between the source and himself, or he can connect himself via other players.

There are two types of connections: public and private. If a player constructs a public link, other players can use this link to get to the source. A private connection can only be used by the player who constructs it.

When constructing a network, players can cooperate in order to reduce costs. We assume that if a group of players cooperate, the players within that coalition construct an optimal public network, which by definition is open for use by other players. Once this optimal public network for the coalition is constructed, the players outside can decide whether or not to connect to the source, using the public network in place, possibly complemented with private connections. The corresponding payoffs to these individual players are the spillovers that result from the formation of this coalition. We call the resulting model a public-private connection (ppc) problem. Note that in principle every coalition can build the public network and hence, $\mathcal{W} = 2^N$.

In section 5 we introduce ppc problems more formally and analyse their corresponding games.

²It is not within the scope of this paper to provide an underlying model from which these coalition values can be derived. We simply give some *ad hoc* numbers to illustrate the concept of spillover game.

Example 3 Consider a group of firms in an oligopolistic market. We assume that these firms play a Cournot game and that the outcome will be the Cournot (Nash) equilibrium of this game.

If the firms cooperate and form a cartel, in general their total profit increases. Indeed, in the extreme case of full cooperation, a *de facto* monopoly will ensue with corresponding monopoly profits. We can also consider the situation in which only a coalition of firms $S \subset N$ decides to form a cartel. Clearly, this will increase the total profit of the firms in S . But not only will the members of S be affected by the cartel. By cooperating, the firms in the cartel influence the market price, and hence, the profits of the other firms. In section 6, we show how these cartel profits and corresponding externalities can be captured in a spillover game.

3 The core

In this section we extend the definition of core for TU games to the class of spillover games. Furthermore, we characterise nonemptiness of the core by means of a balancedness property.

A payoff vector $x \in \mathbb{R}^N$ belongs to the S -core if for every coalition, the total payoff to the members of that coalition exceeds its value. So, for $S \in \mathcal{W}$ we define the S -core by

$$C_S(\mathcal{G}) = \{x \in \mathbb{R}^N \mid \sum_{i \in S} x_i = v(S), x_{N \setminus S} = z^S, \forall T \subset N : \sum_{i \in T} x_i \geq v(T)\},$$

or, equivalently,

$$C_S(\mathcal{G}) = \{x \in \mathbb{R}_+^N \mid \sum_{i \in S} x_i = v(S), x_{N \setminus S} = z^S, \forall T \in \mathcal{W} : \sum_{i \in T} x_i \geq v(T)\}.$$

An allocation in the S -core is stable in the sense that there is no other winning coalition T that objects to the proposed allocation on the basis of it being able to obtain more if it cooperates. The *core* of \mathcal{G} consists of all undominated payoff vectors in the union of all S -cores, so

$$C(\mathcal{G}) = \text{undom}(\bigcup_{S \in \mathcal{W}} C_S(\mathcal{G})),$$

where $\text{undom}(A) = \{x \in A \mid \neg \exists y \in A : y \succ x\}$.

For TU games, Bondareva (1993) and Shapley (1967) characterised nonemptiness of the core by means of the concept of *balancedness*. We establish a similar result for the class of \mathcal{W} -stable spillover games. A game $\mathcal{G} = (N, \mathcal{W}, v, z)$ is called \mathcal{W} -stable if

$$S, T \in \mathcal{W}, S \cap T = \emptyset \Rightarrow \begin{cases} \sum_{i \in T} z_i^S \geq v(T), \\ \sum_{i \in S} z_i^T \geq v(S). \end{cases}$$

The idea behind \mathcal{W} -stability is that there can exist no two disjoint winning coalitions with positive spillovers. For, if two such coalitions were present, the game would have no stable outcome in the sense that both these coalitions would want to form. Note that positive spillover games and spillover games with N -proper \mathcal{W} belong to the class of \mathcal{W} -stable games.

For $S \subset N$, we define e^S to be the vector in \mathbb{R}^N with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \notin S$. A map $\lambda : \mathcal{W} \rightarrow \mathbb{R}_+$ is called S -subbalanced if

$$\sum_{T \in \mathcal{W}} \lambda(T) e_S^T \leq e_S^S.$$

We denote the set of all such S -subbalanced mappings by \mathcal{B}^S .

A game $\mathcal{G} = (N, \mathcal{W}, v, z)$ is S -subbalanced if for all S -subbalanced $\lambda : \mathcal{W} \rightarrow \mathbb{R}_+$ it holds that

$$\sum_{T \in \mathcal{W}} \lambda(T) \left[v(T) - \sum_{i \in (N \setminus S) \cap T} z_i^S \right] \leq v(S).$$

Suppose winning coalition S forms, giving its members a total payoff of $v(S)$. Next, consider a winning coalition T and consider the situation where T forms. The payoff to T would then be $v(T)$, but some of its members would have to forego the spillovers resulting from the formation of S . So, after subtracting these opportunity costs, the net payoff to T equals the expression inside the brackets. A game is S -subbalanced if dividing the net payoffs of all winning coalitions T in an S -subbalanced way yields a lower payoff than $v(S)$.

Theorem 1 *Let $\mathcal{G} = (N, \mathcal{W}, v, z)$ be a \mathcal{W} -stable spillover game. Then $C(\mathcal{G}) \neq \emptyset$ if and only if there exists an $S \in \mathcal{W}$ such that \mathcal{G} is S -subbalanced.*

Proof. See appendix.

4 Government situations

In this section, we analyse government situations and introduce a power index that can be used to indicate the relative power of the parties involved. On the basis of Example 1 we introduce the concept of marginal vector for spillover games, which we use to define a power index that is reminiscent of the Shapley value for TU games. Contrary to its TU counterpart, strategic considerations play an important role in our definition of marginal vector.

Example 4 *Recall the government situation in Example 1. To construct a marginal vector, assume that first the largest party, LIB, enters. Since this party on itself*

is not winning, its marginal contribution is zero. To keep things simple, we assume that parties always join if the coalition in place is not yet winning. Hence, the second largest party, *SOC*, joins, creating a winning coalition. Its payoff equals the marginal contribution to the existing coalition, which equals $10-0=10$. Next, the third largest, *CD* has the choice whether to join or not. If it joins, its marginal contribution is $18-10=8$. If it does not join, the worst that can happen is that coalition $\{COM, SOC, LIB\}$ eventually cooperates, giving *CD* a payoff (spillover) of 6. Hence, *CD* joins the existing coalition. Finally, *COM* decides not to join, giving it a spillover of 1 rather than the marginal contribution of -2. So, the resulting coalition will be $\{SOC, CD, LIB\}$ with payoff 1 to *COM*, 10 to *SOC*, 8 to *CD* and 0 to *LIB*.

The procedure described in the previous example resembles the well-known concept of marginal vector for TU games. The crucial difference, however, is that contrary to the TU case, in our context players do not *have to* join the existing coalition. As long as there is a winning coalition in place and the worst that can happen if a player does not join is better than joining, that player has the option to stay outside.

In order to define the concept of marginal vector, we need to introduce some more notation. An *ordering* on the player set N is a bijection $\sigma : \{1, \dots, n\} \rightarrow N$, where $\sigma(k) = i$ means that player $i \in N$ is at position k . The set of all $n!$ orderings on N is denoted by $\Pi(N)$. The set of predecessors of player $i \in N$ according to ordering $\sigma \in \Pi(N)$ is denoted by $P_i^\sigma = \{j \in N \mid \sigma^{-1}(j) < \sigma^{-1}(i)\}$.

Let (N, \mathcal{W}, v, z) be a spillover game. The *marginal vector corresponding to σ* , $\sigma \in \Pi(N)$, denoted by $M^\sigma(N, \mathcal{W}, v, z)$, is defined recursively. By S_k^σ we denote the current coalition after the first k players have entered and we initialise $S_0^\sigma = \emptyset$. Let $k \in \{1, \dots, n\}$. We assume that player $i = \sigma(k)$ has to join the coalition in place, S_{k-1}^σ , if this coalition is not yet winning. Otherwise, he has to choose between joining and staying out. As a result of monotonicity of \mathcal{W} , once a winning coalition is in place, a winning coalition will result regardless of whether the next player joins or not. The minimum payoff to player $i = \sigma(k)$ if he chooses not to join the winning coalition S_{k-1}^σ equals

$$m_i^\sigma = \min_{T \subset N \setminus \{i\} : T \cap P_i^\sigma = S_{k-1}^\sigma} z_i^T.$$

If he does join, his marginal contribution equals

$$c_i^\sigma = v(S_{k-1}^\sigma \cup \{i\}) - v(S_{k-1}^\sigma).$$

If player i has the choice, he decides not to join S_{k-1}^σ if the worst that can happen to i if he stays outside, m_i^σ , is better than his marginal contribution c_i^σ . So,

$$S_k^\sigma = \begin{cases} S_{k-1}^\sigma & \text{if } S_{k-1}^\sigma \in \mathcal{W} \text{ and } m_i^\sigma > c_i^\sigma, \\ S_{k-1}^\sigma \cup \{i\} & \text{otherwise} \end{cases}$$

and

$$M_i^\sigma(N, \mathcal{W}, v, z) = \begin{cases} z_i^{S_k^\sigma} & \text{if } S_{k-1}^\sigma \in \mathcal{W} \text{ and } m_i^\sigma > c_i^\sigma, \\ c_i^\sigma & \text{otherwise.} \end{cases}$$

According to this procedure, the coalition $S^\sigma = S_n^\sigma$ eventually results and in the corresponding marginal vector, $v(S^\sigma)$ is divided among the members of S^σ and the players in $N \setminus S^\sigma$ get their corresponding spillovers.

The solution that is computed in Example 4 is the marginal vector that corresponds to the ordering based on the shares of the seats. Of course, this procedure can be performed with all orderings on the parties, each leading to a marginal vector. The *Shapley value* (cf. ?)) is defined as the average of these marginal vectors:

$$\Phi(N, \mathcal{W}, v, z) = \frac{1}{n!} \sum_{\sigma \in \Pi(N)} M^\sigma(N, \mathcal{W}, v, z).$$

The Shapley value can be interpreted as an *expected* vector of power indices for parties if the orderings on the players are equally likely. The total power according to different marginal vectors need not be the same. Contrary to each marginal vector separately, the Shapley value is not “supported” by a single coalition.

Example 5 Recall from Example 4 that the marginal vector of the game in Example 1 corresponding to the order $\sigma = (LIB, SOC, CD, COM)$ equals $M^\sigma = (1, 10, 8, 0)$, with resulting coalition $\{SOC, CD, LIB\}$. If we take the order $\tau = (CD, COM, LIB, SOC)$, we obtain the marginal vector $M^\tau = (0, 4, 0, 14)$ with corresponding coalition $\{COM, CD, LIB\}$. Note that $\sum_{i \in N} M_i^\sigma \neq \sum_{i \in N} M_i^\tau$.

Computing all marginal vectors and taking the average yields the Shapley value:

$$\Phi(N, \mathcal{W}, v, z) = \frac{1}{24}(24, 140, 172, 116).$$

It is readily seen that there exists no order on the parties such that the corresponding marginal vector coincides with the Shapley value and hence, there is no single coalition giving rise to this solution.

The procedure presented in the definition of marginal vector should not be viewed as a description of how governing coalitions are or should be formed. Rather, these marginal vectors are an indication of what could happen and through the Shapley value, they provide an insight into the relative power of the players.

The strategic element in our definition of marginal vector is that a player can choose not to join when it is in his interest to stay separate. We assume that players are cautious in that they only decide not to join when the worst that can happen when doing so is better than the payoff if they join.³ This strategic element can be extended in several ways. For example, one can assume that the players play a sequential move extensive form game and the resulting marginal vector is the payoff vector corresponding to a subgame perfect equilibrium. One would then have to make some additional assumptions about what happens when indifferences occur or when there are multiple equilibria.

5 Public-private connection problems

In many allocation decisions resulting from Operations Research (OR) problems, spillovers occur naturally. In this section, we analyse public-private connection (ppc) problems as described in Example 2. We address two main questions: which coalition will cooperate and how should the value of this coalition be divided among its members?

Before formally introducing ppc problems, we start with an example.

Example 6 Consider the ppc problem depicted in Figure 1, where $*$ is the source, the bold numbers indicate the players, the numbers between parentheses represent the benefits if the players are connected to the source and the numbers on the edges are the corresponding construction costs.

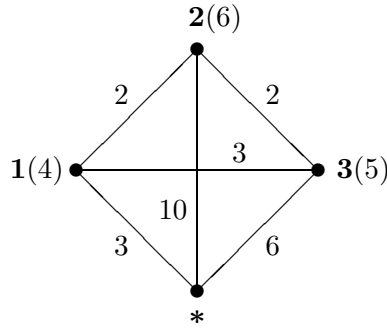


Figure 1: A ppc problem

First, consider the grand coalition. The best this coalition can do is to build a public network connecting all players to the source, creating links $\{*, 1\}$, $\{1, 2\}$ and

³This is quite standard practice in cooperative game theory. Usually, a noncooperative game is turned into a TU game by assigning values to coalitions based on the maximin principle, ie, by assuming that the players in a coalition maximise their payoff given that the other players try to minimise this payoff.

$\{2, 3\}$. The net payoff equals $4 + 6 + 5 - (3 + 2 + 2) = 8$.

Next, consider coalition $\{2\}$ ⁴. It is optimal for this coalition to create $\{*, 1\}$ and $\{1, 2\}$, giving player 2 a payoff of $6 - (2 + 3) = 1$. The construction of these public links results in spillovers for players 1 and 3. Player 1 can use the public network and does not have to create an extra private link, so his spillover equals his benefit of 4. Player 3 can also use the public network, complemented with the private connection $\{2, 3\}$, giving him a spillover of $5 - 2 = 3$.

Next, consider $\{3\}$. Since every path to the source is more expensive than his benefit, player 3 will not construct a network at all, giving him a payoff of 0. Player 1 then has to construct a private link $\{*, 1\}$ with spillover 1 and player 2, who cannot use 1's private link, will have to construct $\{*, 1\}$ and $\{1, 2\}$ privately, giving him a spillover of 1 as well.

Doing this for every possible coalition, we obtain a spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ with $N = \{1, 2, 3\}$, $\mathcal{W} = 2^N$ and the following payoffs:

S	\emptyset	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	N
$v(S)$	0	1	1	0	5	3	4	8
z^S	(1, 1, 0)	(4, 2)	(4, 3)	(1, 1)	3	4	4	

A *public-private connection* or *ppc problem* is a triple $(N, *, b, c)$, where $N = \{1, \dots, n\}$ is a set of agents, $*$ is a source, $b : N \rightarrow \mathbb{R}_+$ is a nonnegative benefit function and $c : E_{N^*} \rightarrow \mathbb{R}_+$ is a nonnegative cost function, where $N^* = N \cup \{*\}$. E_S is defined as the set of all edges between pairs of elements of $S \subset N^*$, so that (S, E_S) is the complete graph on S :

$$E_S = \{\{i, j\} \mid i, j \in S, i \neq j\}.$$

$b(i)$ represents the benefits if player $i \in N$ is connected to $*$ and $c(\{i, j\})$ represents the costs if a link between $i \in N^*$ and $j \in N^*$ is formed.

Links can be created either publicly or privately, as described in Example 2. To avoid unnecessary diversions, we simply assume that the optimal public network for each coalition is unique.

A network of edges is a set $K \subset E_{N^*}$. By $N(K) \subset N$ we denote the set of players that are connected to the source in network K .

A ppc problem $(N, *, b, c)$ gives rise to a *public-private connection* or *ppc game* (N, \mathcal{W}, v, z) with $\mathcal{W} = 2^N$,

$$v(S) = \max_{K \subset E_{N^*}} \left\{ \sum_{i \in S \cap N(K)} b(i) - \sum_{k \in K} c(k) \right\} \quad (2)$$

⁴It may seem strange that a single player or even the empty coalition can build a public network. For the sake of expositional clarity, we do not *a priori* exclude this possibility.

for all $S \subset N$ and

$$z_i^S = \max_{L \subset E_{N^*} \setminus K_S} \{b(i)I_{N(K_S \cup L)}(i) - \sum_{\ell \in L} c(\ell)\},$$

for all $S \subset N, i \in N \setminus S$, where K_S denotes the unique network K that maximises (2), and $I_A(i)$ equals 1 if $i \in A$ and 0 if $i \notin A$.

Although players outside S can use the public network created by S , the spillovers need not be positive. This is caused by the assumption that only the players within the coalition that eventually builds the public network can cooperate, whereas the players outside can only build private links. As a result, the costs of a particular connection may have to be paid more than once by the players outside the coalition and consequently, they are worse off than when they cooperate.

A spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ is called *superadditive* if

$$v(T) \geq v(S) + \sum_{i \in T \setminus S} z_i^S$$

for all $S \subset T \subset N$. If a game is superadditive, then it is beneficial to form a large coalition: the payoff to S and $T \setminus S$ is larger if these coalitions merge rather than stay separate. Note that if spillovers are positive, this condition is stronger than the TU definition $v(T) \geq v(S) + v(T \setminus S)$, ie, if spillovers are positive, the coalitions have a bigger incentive not to merge.

\mathcal{G} is *convex* if

$$v(S \cup T) + \sum_{i \in S \cap T} z_i^{S \setminus T} \geq v(S) + \sum_{i \in T} z_i^{S \setminus T}$$

for all $S, T \subset N$, or equivalently,

$$v(T \cup U) - \sum_{j \in T} z_j^U \geq v(S \cup U) - \sum_{j \in S} z_j^U \quad (3)$$

for all $U \subset N, S \subset T \subset N \setminus U$. Convexity can be interpreted in terms of increasing marginal contributions: if a large coalition T decides to join U , then its marginal contribution, being the value of the resulting coalition minus the opportunity costs of staying separate, is larger than the marginal contribution (to U) of a smaller coalition S .⁵ It is readily verified that every convex spillover game is superadditive.

Public-private connection games are superadditive, as is shown in the following proposition.

⁵Note that contrary to its TU analogue, requiring (3) only for $U \subset N$ with $|U| = 1$ is strictly weaker.

Proposition 1 *Let $(N, *, b, c)$ be a ppc problem. Then the corresponding game (N, \mathcal{W}, v, z) is superadditive.*

Proof. Let $S \subset T \subset N$. Let K_S be the optimal public network for S and for all $i \in T \setminus S$, let $L_i^{N \setminus S}$ be the optimal private network for i , given that K_S is present. Define $\bar{K} = K_S \cup \bigcup_{i \in T \setminus S} L_i^{N \setminus S}$. Then

$$\begin{aligned}
v(T) &= \max_{K \subset E_{N^*}} \left\{ \sum_{i \in T \cap N(K)} b(i) - \sum_{k \in K} c(k) \right\} \\
&\geq \sum_{i \in T \cap N(\bar{K})} b(i) - \sum_{k \in \bar{K}} c(k) \\
&= \sum_{i \in S \cap N(\bar{K})} b(i) - \sum_{k \in K_S} c(k) + \sum_{i \in (T \setminus S) \cap N(\bar{K})} b(i) - \sum_{k \in \bar{K} \setminus K_S} c(k) \\
&\geq v(S) + \sum_{i \in T \setminus S} b(i) I_{N(\bar{K})}(i) - \sum_{k \in \bar{K} \setminus K_S} c(k) \\
&\geq v(S) + \sum_{i \in T \setminus S} \left[b(i) I_{N(K_S \cup L_i^{N \setminus S})}(i) - \sum_{\ell \in L_i^{N \setminus S}} c(\ell) \right] \\
&= v(S) + \sum_{i \in T \setminus S} z_i^S.
\end{aligned}$$

□ Although public-private connection games are superadditive, they need not be convex, as is illustrated in the following example.

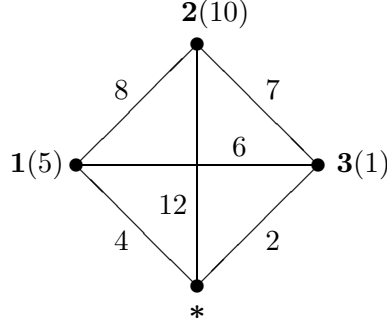


Figure 2: A ppc problem

Example 7 *Consider the ppc problem depicted in Figure 2. Let $S = \{1\}$, $T = \{1, 3\}$*

and $U = \{2\}$. Then for the corresponding game (N, \mathcal{W}, v, z) we have

$$\begin{aligned}
v(T \cup U) - \sum_{j \in T} z_j^U &= v(N) - z_1^{\{2\}} - z_3^{\{2\}} \\
&= 3 - 1 - 1 \\
&< 3 - 1 \\
&= v(\{1, 2\}) - z_1^{\{2\}} \\
&= v(S \cup U) - \sum_{j \in S} z_j^U.
\end{aligned}$$

Hence, this game is not convex.

Let us return to the ppc problem in Example 6. To find a suitable solution for this problem, we first consider the core of the corresponding ppc game \mathcal{G} . The S -cores are given in the following table, where $\text{Conv}(A) = \{x \in \mathbb{R}^N \mid \exists t \in \mathbb{N} \exists x^1, \dots, x^t \in A \exists (\lambda_1, \dots, \lambda_t) \in \Delta^t : x = \sum_{i=1}^t \lambda_i x^i\}$ is the convex hull of $A \subset \mathbb{R}^N$.

S	$C_S(\mathcal{G})$
\emptyset	\emptyset
$\{1\}$	$\{(1, 4, 2)\}$
$\{2\}$	$\{(4, 1, 3)\}$
$\{3\}$	\emptyset
$\{1, 2\}$	$\text{Conv}(\{(4, 1, 3), (1, 4, 3)\})$
$\{1, 3\}$	$\text{Conv}(\{(3, 4, 0), (1, 4, 2)\})$
$\{2, 3\}$	$\text{Conv}(\{(4, 4, 0), (4, 1, 3)\})$
N	$\text{Conv}(\{(4, 1, 3), (4, 4, 0), (3, 5, 0), (1, 5, 2), (1, 4, 3)\})$

Since the N -core (weakly) dominates all the other cores, we have $C(\mathcal{G}) = C_N(\mathcal{G})$. Note that there are some core elements that are supported by other coalitions as well, all of which contain player 2. The core element $(4, 1, 3)$ is even supported by every coalition containing player 2.

In Figure 3 we depict the four S -cores that yield core elements and (therefore) lie in the hyperplane with total payoff 8. The payoff to player 1 is in normal typeface, the payoff to player 2 is italic and the payoff to player 3 is bold. The N -core $C_N(\mathcal{G})$ is the shaded pentagon, $C_{\{1,2\}}$ is the line segment with the triangles, $C_{\{2,3\}}$ is the line segment with the stars and $C_{\{2\}}$ is the point $(4, 1, 3)$.

To solve the ppc problem, suppose for the moment that all players cooperate. We have already seen that it is optimal for the grand coalition to connect all its members to the source. Since the benefits of a coalition do not depend on the shape of the network that is formed as long as everyone is connected, the optimal network

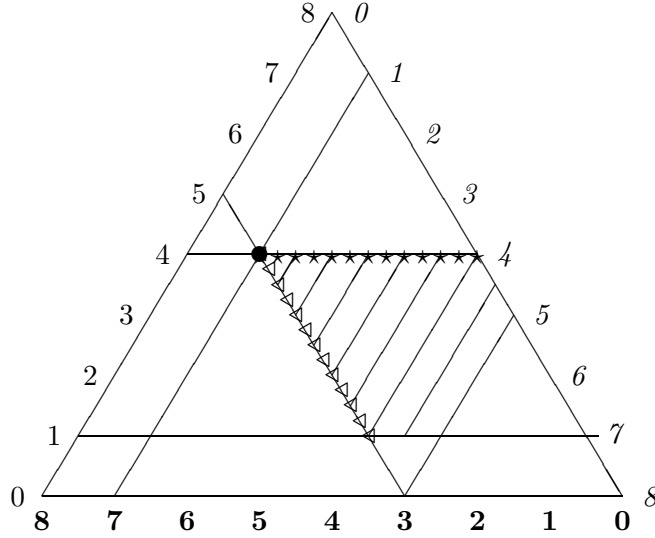


Figure 3: The core of \mathcal{G}

in this ppc problem, $\{\{*, 1\}, \{1, 2\}, \{2, 3\}\}$, is actually a *minimum cost spanning tree* (Claus and Kleitman (1973)). In this context, Bird (1976) proposed that each player pays the costs of the (unique) link that is adjacent to him and lies on the path between him and the source. So, one way to solve a ppc problem is to assume that construction costs are divided using Bird's rule and everyone gets his own benefit. According to this Bird-like procedure, player 1 receives $4 - 3 = 1$, player 2 gets $6 - 2 = 4$ and player 3 gets $5 - 2 = 3$. This yields the core element $(1, 4, 3)$ as solution.

This procedure, however, has some elementary flaws. The nice properties of the Bird rule for mcst problems follow from the assumption that all players *have to* cooperate and connect to the source. Moreover, this rule does not take the spillovers into account. The strategic option of players not to participate in a coalition undermines the Bird approach. Player 1 will never agree to the proposed payoff vector $(1, 4, 3)$, since he will be better off leaving the grand coalition, which will lead to a payoff (spillover) of 4. Knowing this, player 3 can argue that he should at least receive 3, his spillover when player 2 forms a coalition on his own. Taking this into account, the payoff vector $(4, 1, 3)$ seems a more reasonable outcome. Because player 2 on his own will build a network that also connects player 1 to the source, the latter player occupies a position of power in this ppc problem, which should somehow be reflected in his payoff.

The payoff vector $(4, 1, 3)$ is a core element of the corresponding ppc game and

is supported by all coalitions containing player 2. This payoff, however, is not acceptable to player 2. He can argue that if he were to refuse to build his optimal public network, it would then be optimal for players 1 and 3 to work together, giving player 2 a spillover of 4.

By considering this kind of strategic threats of the players not to cooperate, any seemingly reasonable proposal can be dismissed. As a result, it is not clear which coalition will eventually emerge and what the corresponding payoffs will be.

This phenomenon of free-riding is well-known in the context of public goods. Although it is socially optimal for all the players to cooperate in order to provide a public good, the players separately have the strategic incentive not to do so.

One way to solve this problem is to apply the Shapley value, as defined in the previous section. In each marginal vector, the strategic aspects mentioned above are taken into account. By averaging over all marginal vectors, some kind of “average” influence of these noncooperative considerations is reflected in the payoff.

Example 8 *Consider the ppc problem of Example 6. The Shapley value equals*

$$\Phi(N, \mathcal{W}, v, z) = \frac{1}{6}(17, 20, 11).$$

As was discussed in the previous section, the definition of marginal vector can be adapted to reflect the level of strategic elements one wants to incorporate in the model.

6 Cartels in oligopolistic markets

In this section we show how to construct a spillover game out of the model of an oligopolistic market that is characterised by Cournot (quantity) competition. The Shapley value that was introduced in Section 4 is used to assess the relative power of firms in the market.

Consider a market with set of firms $N = \{1, \dots, n\}$. The inverse demand function is given by $P : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$. The production technology of each firm $i \in N$ is represented by the cost function $C_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$. The profit function for firm i , $\pi_i : \mathbb{R}_+^N \rightarrow \mathbb{R}$, is defined by

$$\pi_i(q_i, q_{-i}) = P(q_i, q_{-i})q_i - C_i(q_i)$$

for all $q \in \mathbb{R}_+^N$, where $q_{-i} = \{q_j\}_{j \in N \setminus \{i\}}$. Simultaneously solving the profit maximisation problem for all firms yields the Cournot-Nash equilibrium profits $\bar{\pi} = (\bar{\pi}^1, \dots, \bar{\pi}^n)$.

This oligopolistic market can be modelled as a spillover game $\mathcal{G} = (N, \mathcal{W}, v, z)$ in the following way. Take $\mathcal{W} = 2^N$. The Cournot-Nash equilibrium corresponds to a situation where $S = \emptyset$ with $v(\emptyset) = 0$ and $z^\emptyset = \bar{\pi}$. Subsequently, let $S = \{i\}$ for some $i \in N$. This still corresponds to a Cournot situation, ie, $v(\{i\}) = \bar{\pi}_i$ and $z^{\{i\}} = \bar{\pi}_{-i}$.

For any coalition S consisting of more than one firm, we assume that there are economies of scale so that the coalition (or cartel) can produce the product at the lowest possible costs. So, if $2 \leq |S| \leq n - 1$, we basically have a Cournot situation with $n - |S| + 1$ firms:

$$\begin{cases} \max_{q^S \in \mathbf{R}_+^S} \{P(q^S, q_{N \setminus S}) \sum_{i \in S} q_i^S - \sum_{i \in S} C_i(q_i)\}, \\ \max_{q_j \in \mathbf{R}_+} \{P(q^S, q_{N \setminus S}) q_j - C_j(q_j)\} \end{cases} \quad \text{for all } j \in N \setminus S.$$

This gives the vector of equilibrium profits $(\bar{\pi}^S, \bar{\pi}^{N \setminus S})$, where $\bar{\pi}^{N \setminus S} = (\bar{\pi}_j)_{j \in N \setminus S}$, leading to the values $v(S) = \bar{\pi}^S$ and $z^S = \bar{\pi}^{N \setminus S}$.

For the grand coalition N , we simply solve the monopoly problem and obtain

$$v(N) = \max_{q^N \in \mathbf{R}_+^N} \left\{ P(q^N) \sum_{i \in N} q_i^N - \sum_{i \in N} C_i(q_i) \right\}.$$

The Shapley value can be used as an indication of the market power that firms have when they cooperate in a cartel. The advantage of using the framework of spillover games for oligopolistic markets is that it incorporates in one model such extreme cases as fully noncooperative competition and fully cooperative collusion. Furthermore, all intermediate cases are included as well. The Shapley value averages out the profits in all these cases, thereby indicating the relative power of firms in the market.

Note, however, that the Shapley value only considers one cartel for each permutation. There is no *a priori* reason why there could not be two or more cartels in one market.

In the remainder of this section, we consider some examples of a market with three firms. The linear inverse demand function is specified by $P(q_1, q_2, q_3) = b - a(q_1 + q_2 + q_3)$, $a > 0$, $b > 0$. It is assumed that the cost functions have constant marginal costs, ie, for firm i it holds that $C_i(q) = c_i q$, $0 < c_1 \leq c_2 \leq c_3$. Additionally, it is assumed that $b \geq 3c_1 - c_2 - c_3$, $b \geq 2c_1 - c_3$, and $b \geq c_1$.⁶ For this spillover game the values for v and z are given in Table 1.

It is not clear *a priori* what will happen in a market like this. In the following example we analyse two situations.

⁶These assumptions ensure that all equilibrium profits are nonnegative.

S	$v(S)$	z^S
\emptyset	0	$(\frac{(b+c_2+c_3-3c_1)^2}{16a}, \frac{(b+c_1+c_3-3c_2)^2}{16a}, \frac{(b+c_1+c_2-3c_3)^2}{16a})$
$\{1\}$	$\frac{(b+c_2+c_3-3c_1)^2}{16a}$	$(\frac{(b+c_1+c_3-3c_2)^2}{16a}, \frac{(b+c_1+c_2-3c_3)^2}{16a})$
$\{2\}$	$\frac{(b+c_1+c_3-3c_2)^2}{16a}$	$(\frac{(b+c_2+c_3-3c_1)^2}{16a}, \frac{(b+c_1+c_2-3c_3)^2}{16a})$
$\{3\}$	$\frac{(b+c_1+c_2-3c_3)^2}{16a}$	$(\frac{(b+c_2+c_3-3c_1)^2}{16a}, \frac{(b+c_1+c_3-3c_2)^2}{16a})$
$\{1, 2\}$	$\frac{(b+c_3-2c_1)^2}{9a}$	$\frac{(b+c_1-2c_3)^2}{9a}$
$\{1, 3\}$	$\frac{(b+c_2-2c_1)^2}{9a}$	$\frac{(b+c_1-2c_2)^2}{9a}$
$\{1, 2\}$	$\frac{(b+c_3-2c_2)^2}{9a}$	$\frac{(b+c_2-2c_1)^2}{9a}$
N	$\frac{(b-c_1)^2}{4a}$	

Table 1: Values for v and z .

Example 9 For $b = 50$, $a = 1$, $c_1 = 5$, $c_2 = 10$, $c_3 = 15$ the Shapley value is approximately given by $\Phi(\mathcal{G}) \approx (280, 147, 80)$. With the same values for b and a , and cost parameters $c_1 = 5$, $c_2 = 6$ and $c_3 = 10$, we have $\Phi(\mathcal{G}) \approx (211, 184, 67)$. Finally, with constant marginal costs $c_1 = c_2 = c_3 = 10$ one obtains $\Phi(\mathcal{G}) = (100, 100, 100)$.

From the previous example one can deduce that apparently the magnitude of the synergy effects of collusion is important. Consider for example the case where firms are symmetric. Take any permutation of N . From Table 1 one can see that the first firm is indifferent between joining and not joining. It was assumed, in defining the Shapley value, that in case of indifference a player joins. The firm that arrives second has to compare its marginal contribution with the worst-case outside option. Joining the coalition means that the third firm gets an equal profit to the two firms in the cartel (since firms are symmetric). If marginal costs equal c , then the marginal contribution of the second firm equals $\frac{7(b-c)^2}{144a} < \frac{(b-c)^2}{16a}$. Hence, the second firm will never join. Then the same holds for the third firm.

In the following example we show the influence of the demand parameter a on the Shapley value.

Example 10 Consider an oligopolistic market with three firms, affine demand and linear costs. The marginal costs for the three firms equal $c_1 = 5$ and $c_2 = c_3 = 10$, respectively. Since firms 2 and 3 have the same cost function, they also have the same payoff according to the Shapley value. We want to investigate the influence of the slope of the demand function on the difference in Shapley value between firms 1 and 2, ie, we want to assess how the cost advantage of firm 1 works through in the Shapley value. The inverse demand function is taken to be $P(Q) = 50 - aQ$ where a is taken from the interval $[\frac{1}{2}, 5]$. As Figure 4 shows, firm 1's advantage decreases with the slope. The stronger demand reacts on price, the less strategic advantage the more efficient firm has.

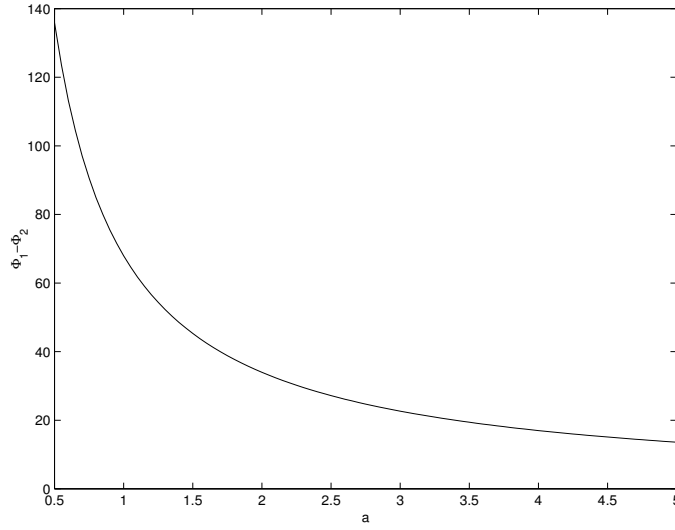


Figure 4: The difference in the Shapley value between firms 1 and 2.

7 Discussion

Public-private connection situations as described in Section 5 are not the only class of OR problems in which spillovers occur. A related phenomenon arises in *travelling salesman situations* (Tamir (1989)). In a travelling salesman situation, there is a graph in which the vertices represent the locations of the players (and the salesman) and the edges represent the roads between them along which the salesman can travel. The problem is to find a cheapest Hamiltonian circuit in this graph, where each edge has a nonnegative cost associated with it.

Also, each subcoalition faces the same problem of finding a cheapest Hamiltonian circuit through the vertices in which the players in this coalition and the salesman are located. This gives rise to a cooperative cost game. As is the case in minimum cost spanning tree situations, however, one does not take into account that there are spillovers involved. If a subcoalition of players decides to work together and invite the salesman to travel to them according to their cheapest tour, the salesman might come near some players outside the coalition, making it cheaper for them to have him come to visit them as well.

In *sequencing situations* (cf. Curiel et al. (1989)), spillovers can also play a role. In a sequencing situation, there is a queue of players waiting to be served. The players in the queue might have different opportunity costs, so moving high-cost players to the front while compensating the low-cost players through side payments can result in a Pareto improvement.

Normally, in such situations, only pairs of players who are adjacent in the queue

are allowed to switch, so that a third player can never suffer. If we use our spillover model, however, this restriction is unnecessary, since the effect of any pairwise switch on the other players can be taken into account explicitly.

In Section 6 we show how an oligopolistic market model can be turned into a spillover game. In economics there are many more cases in which one wants to analyse an essentially noncooperative situation where cooperative considerations play a role. A well-known example from the literature concerns environmental externalities. We refer to the contributions of Chander and Tulkens (1997) and Helm (2001), who analyse a TU game that is obtained in a standard (minmax) way from an economic situation with externalities. They show that a certain equilibrium of the underlying game is an element of the core of the corresponding TU game. However, what these papers do not take explicitly into account is the possibility that firms might benefit from an agreement between their competitors to internalise the externalities and thence opt not to cooperate. This in turn threatens the stability of the agreement between the other firms.

Crucial in the aforementioned externality example is that there is a free-rider effect created by a group of firms that cooperates on the remaining firms. To find a solution that can deal with these effects, they have to be reflected in the model.

Spillover games are tailor-made to achieve this and therefore we think that this class of games can be used in many economic situations. Notably in cases with some sort of market failure (eg, externalities, public goods) where cooperation between economic agents is needed – or even desirable – to mend the failure, but where cooperation need not be enforceable. If cooperation is enforceable, the TU model provides an adequate way to share the benefits/costs related to the market failure. If this is not possible, like for example in many public goods situations, free-rider effects arise and our spillover model is more appropriate.

Appendix

Proof of Theorem 1: Let $S \in \mathcal{W}$. Then

$$\begin{aligned}
C_S(\mathcal{G}) \neq \emptyset &\Leftrightarrow \{x \in \mathbb{R}_+^N \mid \sum_{i \in S} x_i = v(S), x_{N \setminus S} = z^S, \forall T \in \mathcal{W} : \sum_{i \in T} x_i \geq v(T)\} \neq \emptyset \\
&\Leftrightarrow v(S) = \min_{x \in \mathbb{R}^N} \left\{ \sum_{i \in S} x_i \mid \forall T \in \mathcal{W} : \sum_{i \in T} x_i \geq v(T), \forall i \in S : x_i \geq 0, x_{N \setminus S} = z^S \right\} \\
&\stackrel{(*)}{\Leftrightarrow} v(S) = \max_{\lambda, \mu, \psi} \left\{ \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \mu_i z_i^S - \sum_{i \in N \setminus S} \psi_i z_i^S \mid \right. \\
&\quad \left. \sum_{i \in S} \mu_i e^{\{i\}} + \sum_{i \in N \setminus S} \mu_i e^{\{i\}} - \sum_{i \in N \setminus S} \psi_i e^{\{i\}} + \sum_{T \in \mathcal{W}} \lambda(T) e^T = e^S, \lambda, \mu, \psi \geq 0 \right\} \\
&\Leftrightarrow v(S) = \max_{\lambda, \mu, \zeta} \left\{ \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \zeta_i z_i^S \mid \right. \\
&\quad \left. \sum_{i \in S} \mu_i e^{\{i\}} + \sum_{i \in N \setminus S} \zeta_i e^{\{i\}} + \sum_{T \in \mathcal{W}} \lambda(T) e^T = e^S, \lambda, \mu \geq 0 \right\} \\
&\Leftrightarrow \forall \lambda \in \mathcal{B}^S : \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \zeta_i z_i^S \leq v(S), \\
&\quad \text{where } \forall i \in N \setminus S : \zeta_i = \sum_{T \in \mathcal{W} : i \in T} \lambda(T) \\
&\Leftrightarrow \forall \lambda \in \mathcal{B}^S : \sum_{T \in \mathcal{W}} \lambda(T) v(T) + \sum_{i \in N \setminus S} \sum_{T \in \mathcal{W} : i \in T} \lambda(T) z_i^S \leq v(S) \\
&\Leftrightarrow \forall \lambda \in \mathcal{B}^S : \sum_{T \in \mathcal{W}} \lambda(T) \left[v(T) - \sum_{i \in (N \setminus S) \cap T} z_i^S \right] \leq v(S)
\end{aligned}$$

The equivalence (*) follows from duality theory. Note that nonemptiness of the primal feasible set follows from \mathcal{W} -stability and that the dual feasible set is always nonempty. Since $C(\mathcal{G}) \neq \emptyset$ if and only if there exists an $S \in \mathcal{W}$ such that $C_S(\mathcal{G}) \neq \emptyset$, the assertion follows. \square

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