Static Optimization

Mathematical Economics (TR381)

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Remark

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1 Basic Concepts

From a mathematical point of view, everything in this course happens in finite dimensional real spaces. Therefore, we can – without loss of generality – restrict ourselves to the topology on $\mathbb{R}^n$, induced by the Euclidian norm.[1]

Definition 1.1 (Euclidian norm). Let $x, y \in \mathbb{R}^n$. The Euclidian norm between x and y, denoted by $\|x - y\|$, equals

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

Definition 1.2 (ε-neighbourhood). Let $a \in \mathbb{R}^n$. The ε-neighbourhood of a, denoted by $U_\varepsilon(a)$ is the set

$$U_\varepsilon(a) = \{x \in \mathbb{R}^n : \|x - a\| < \varepsilon\}.$$

Using this definition we can define the concept of an open set.

Definition 1.3 (open set; closed set). A set $A \subseteq \mathbb{R}^n$ is open if for every $a \in A$, there exists an $\varepsilon_a > 0$, such that $U_{\varepsilon_a}(a) \subset A$. A set $A \subseteq \mathbb{R}^n$ is closed if A^c is open.

Recall that a set may be neither open nor closed. Take, for example, the interval $A = (0, 1]$. For every $\varepsilon > 0$ it holds that $U_\varepsilon(1) \nsubseteq A$. Also, for $0 \in A^c$, $U_\varepsilon(0) \nsubseteq A^c$. So, neither A nor A^c are open.

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[1] Theorem 6.27 in David Wilkins's notes.

Definition 1.4 (interior point; boundary point)**.** Let $A \subseteq \mathbb{R}^n$. A point $a \in A$ is an interior point of A if there exists an ε-neighbourhood $U_\varepsilon(a)$, such that $U_\varepsilon(a) \subset A$. A point $a \in A$ is a boundary point of A if for all $\varepsilon > 0$, it holds that $U_\varepsilon(a) \nsubseteq A$.

The set of interior points of A is denoted by $int(A)$, whereas the set of boundary points of A is denoted by $\partial(A)$.

In this part of the course we are concerned with optimizing functions. Therefore, we first need to recall the notions of maximum and minimum.

Definition 1.5 (maximum; minimum)**.** Let $A \subseteq \mathbb{R}^n$. The function $f : A \to \mathbb{R}$ attains a

1. global maximum at $a \in A$ if $\forall_{x \in A} : f(x) \leq f(a)$,
2. global minimum at $a \in A$ if $\forall_{x \in A} : f(x) \geq f(a)$,
3. local maximum at $a \in A$ if $\exists_{\varepsilon > 0} \forall_{x \in U_\varepsilon(a)} : f(x) \leq f(a)$,
4. local minimum at $a \in A$ if $\exists_{\varepsilon > 0} \forall_{x \in U_\varepsilon(a)} : f(x) \geq f(a)$.

A maximum/minimum is called *strict* if the inequalities in the above definition are strict. In optimisation theory it is often assumed that functions are differentiable “as often as needed”. We say that $f(\cdot)$ is a C^k function if f has continuous partial derivatives of up to and including order k.

The partial derivative of f with respect to x_i is denoted by $\frac{\partial f}{\partial x_i}$ or $D_i f$. The second order partial derivative of f with respect to x_i and x_j is denoted by $\frac{\partial^2 f}{\partial x_i \partial x_j}$, or $D_{ij} f$.

Definition 1.6 (Jacobian; Hessian)**.** Let $A \subseteq \mathbb{R}^n$ be a set and let $f : A \to \mathbb{R}$ be a C^2 function. The (row-) vector of first order partial derivatives

$$Df = (D_1 f, \dots, D_2 f),$$

is called the Jacobian (matrix) of f. The matrix of second order partial derivatives

$$Hf = \begin{bmatrix} D_{11} f & \cdots & D_{1n} f \\ \vdots & \ddots & \vdots \\ D_{n1} f & \cdots & D_{nn} f \end{bmatrix},$$

is called the Hessian (matrix) of f.

Sometimes we use the transpose of the Jacobian, a (column) vector, which we call the *gradient*:

$$\nabla f = Df^\top.$$

The order of differentiation does not matter:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

In other words, the Hessian is a symmetric matrix.

In determining whether a "candidate optimum" is a maximum or a minimum, it will turn out that the structure of the Hessian plays an important part.

Definition 1.7 (positive (semi) definite)**.** Let A be a symmetric $n \times n$ matrix. Then A is positive (semi) definite if $x^\top Ax > (\geq)0$, for all $x \in \mathbb{R}^n$.

Property 1.1. *Let A be a symmetric $n \times n$ matrix. The following properties are equivalent.*

1. *A is positive (semi) definite.*

2. *All eigenvalues λ_i of A satisfy $\lambda_i > (\geq)0$.*

3. *All principal submatrices[2] of A have a positive (non-negative) determinant.*

Definition 1.8 (negative (semi) definite)**.** Let A be a symmetric $n \times n$ matrix. Then A is negative (semi) definite if $x^\top Ax < (\leq)0$, for all $x \in \mathbb{R}^n$.

Property 1.2. *Let A be a symmetric $n \times n$ matrix. The following properties are equivalent.*

1. *A is negative (semi) definite.*

2. *All eigenvalues λ_i of A satisfy $\lambda_i < (\leq)0$.*

3. *All principal submatrices of $-A$ have a positive (non-negative) determinant.*

We will, therefore, often need to compute determinants of matrices. Some results that will help us follow.

Theorem 1.1. *If A is a square diagonal matrix $A = [a_{ij}]_{i,j}$, then* $\det(A) = a_{11} \cdots a_{nn}$.

Theorem 1.2. *If A is a matrix, then* $\det(A^\top) = \det(A)$.

Theorem 1.3. *If A is a square matrix, then:*

1. *if a multiple of one row of A is added to another row to produce a matrix B, the* $\det(B) = \det(A)$*;*

[2]The k-th principal submatrix of A is the matrix constructed by taking the first k rows and the first k columns from A.

2. *if two rows are interchanged to produce B, then* $\det(B) = -\det(A)$*;*

3. *if one row of A is multiplied by α to produce B, then* $\det(B) = \alpha \cdot \det(A)$*.*

We will also encounter functions mapping sets into higher dimensional Euclidian spaces. So, if $g : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function with typical element $g(x) = (g_1(x), \ldots, g_m(x))$, then the Jacobian matrix – denoted by Dg – is given by

$$Dg = \begin{bmatrix} D_1 g_1 & \cdots & D_n g_1 \\ \vdots & \ddots & \vdots \\ D_1 g_m & \cdots & D_n g_m \end{bmatrix}.$$

In other words, we stack all the Jacobian (row) vectors of the coordinate functions in one matrix. Finally, two important properties.

Property 1.3 (chain rule)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^k \to \mathbb{R}^n$ be functions and let $a \in \mathbb{R}^k$ be such that g is differentiable in a and f is differentiable in $g(a)$. Then the function $f \circ g : \mathbb{R}^k \to \mathbb{R}^m$ is differentiable in a and $D(f \circ g)(a) = Df(g(a))Dg(a)$.*

Property 1.4 (2^{nd} order Taylor polynomial)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^2 function and let $c \in \mathbb{R}^n$. Then for every $x \in \mathbb{R}^n$, there exists $\xi \in \mathbb{R}^n$, such that*

$$f(x) = f(c) + Df(c)(x - c) + \frac{1}{2}(x - c)^\top Hf(\xi)(x - c).$$

2 Optimization of a Function on $\mathbb{R}$

The material in this section is a summary of the results that you have encountered (I assume) in Analysis 1. The central problem is as follows.

Problem 1. Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$. Find the location and type of the optima of f on I.

The following elementary theorem reduces the number of possible candidates substantially.

Theorem 2.1 (first-order condition)**.** *Let $a \in int(I)$. If a is an optimum location for f and if $f'(a)$ exists, then $f'(a) = 0$.*

So, the only possible interior points where f' exists that could qualify as optima locations, are the ones for which $f'(a) = 0$. These points are called the *stationary points* of f. The possible locations of optima can now be found as follows.

Algorithm 1 (finding candidate optima)**.** *1. Determine the stationary points of f;*

2. *Determine the points where f is not differentiable;*

3. *Determine the boundary points of I.*

The next task, of course, is to determine which of these candidates (if any) represents an optimum location. Two results are important here.

Theorem 2.2. *Let $a \in int(I)$ and let f be differentiable in a neighbourhood $U_\varepsilon(a)$ and let $f'(a) = 0$. If*

$$\forall_{x \in U_\varepsilon(a): x<a} : f'(x) > (<) 0, \quad \textit{and} \quad \forall_{x \in U_\varepsilon(a): x>a} : f'(x) < (>) 0,$$

then $f(a)$ is a strict local maximum (minimum) of f.

Theorem 2.3 (second-order condition)**.** *Let $a \in int(I)$ and let f be differentiable on I, such that $f''(a)$ exists. If $f'(a) = 0$ and $f''(a) < (>) 0$, then $f(a)$ is a strict local maximum (minimum) of f.*

Note that Theorems 2.2 and 2.3 do not say anything about boundary points or non-differentiable points. Those have to be studied in isolation.

Example 2.1. Consider the functions $f_1(x) = x^4$, $f_2(x) = -x^4$, and $f_3(x) = x^3$. For each of these we have $f_i'(x) = 0$. Furthermore, $f_i''(0) = 0$. So, Theorem 2.3 does not help us. However, from Theorem 2.2 it immediately follows that $f_i(0) = 0$ is a local maximum for $i = 1$, a local minimum for $i = 2$, and neither for $i = 3$.

Finally, we address the question whether one can say something about the global character of optima. In general, that is a difficult question to answer, but there are two cases where something can be said. Firstly, if f is differentiable on I, then Theorem 2.2, together with an investigation of the closure of $int(I)$ gives a full answer. Secondly, if I is compact we have the following result.

Theorem 2.4 (Weierstrass)**.** *Let I be compact and non-empty, and let f be continuous on I. The f has a global maximum and a global minimum.*

Example 2.2. Let $f : [-1, 3] \to \mathbb{R}$, where

$$f(x) = \frac{1}{4}x^4 - \frac{5}{6}x^3 + \frac{1}{2}x^2 - 1.$$

1. The stationary points are

$$\begin{aligned} & f'(x) = 0 \\ \iff & x^3 - \tfrac{5}{2}x^2 + x = x(x^2 - \tfrac{5}{2}x + 1) = x(x-2)(x - \tfrac{1}{2}) = 0 \\ \iff & x = 0 \vee x = 2 \vee x = \tfrac{1}{2}. \end{aligned}$$

2. Together with the boundary points this gives the candidate optima locations

$$x = -1 \vee x = 0 \vee x = 2 \vee x = \tfrac{1}{2} \vee x = 3.$$

3. These locations have the values

$$f(-1) = \tfrac{7}{12}, f(0) = -1, f(1/2) = -1 + \tfrac{7}{192}, f(2) = -\tfrac{5}{3}, f(3) = \tfrac{5}{4}.$$

So, the global minimum is $f(2) = -5/3$ and the global maximum is $f(3) = 5/4$.

3 Convexity

Theorem 2.3 gives a sufficient condition for the nature of a stationary point. From a geometric point of view this theorem tells us that local concavity (convexity) is a sufficient condition for a stationary point to be a local maximum (minimum). Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is concave (convex) if the line connecting any two points on the graph of f lies entirely under (above) the graph itself (see Figure 1)

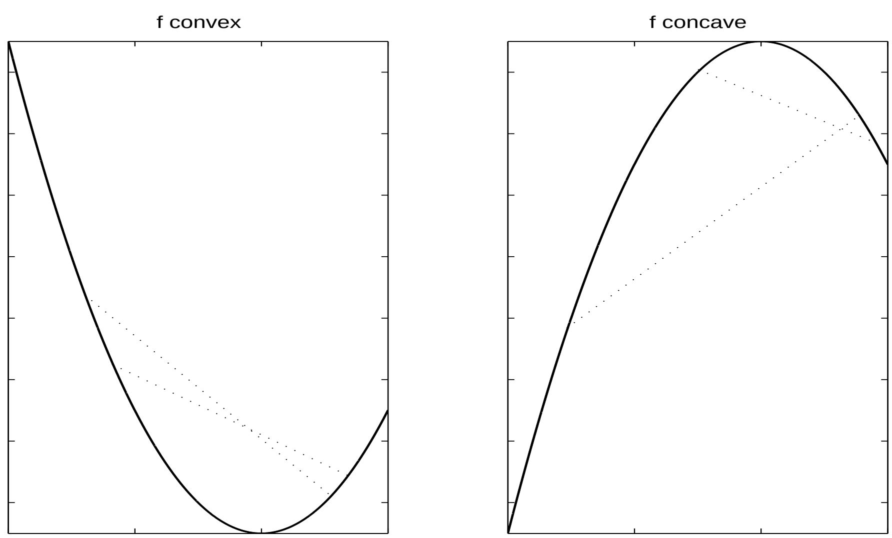

Figure 1: A convex (left-panel) and concave (right-panel) function.

This idea can be generalized to $\mathbb{R}^n$.

Definition 3.1 (convex set)**.** A set $A \subset \mathbb{R}^n$ is convex if for all $x, y \in A$ and $\lambda \in [0, 1]$ it holds that $\lambda x + (1 - \lambda)y \in A$.

The point $\lambda x + (1 - \lambda)y \in A$ is called a *convex combination* and is just a straight line between x and y.

Definition 3.2 (concave/convex function)**.** Let $f : A \to \mathbb{R}$, with $A \subset \mathbb{R}^n$ convex. The function f is concave (convex) if for all $x, y \in A$ and all $\lambda \in [0, 1]$ it holds that

$$f(\lambda x + (1-\lambda)y) \geq (\leq) \lambda f(x) + (1-\lambda) f(y).$$

Note that convexity of A ensures that $f(\lambda x + (1-\lambda)y)$ exists. f is called strictly concave (convex) if the inequality is strict. Also, f is (strict) convex iff $-f$ is (strict) concave.

From Figure 1 you can see that another way to characterize concavity is by using tangent lines. You can see that f is concave (convex) if for every point a, the graph of f lies entirely below (above) the tangent line through a.

Lemma 3.1. *Let $f : A \to \mathbb{R}$ be C^1, with $A \subset \mathbb{R}^n$ open and convex. Then f is concave (convex) if and only if for all $x, a \in A$, it holds that*

$$f(x) - f(a) \leq (\geq) Df(a)(x-a).$$

f is strictly concave (convex) if the inequality is strict for all $x \neq a$.

We can now generalize Theorem 2.3.

Theorem 3.1. *Let $f : A \to \mathbb{R}$ be C^2, with $A \subset \mathbb{R}^n$ open and convex. Then f is concave (convex) if and only if for all $x \in A$, it holds that $Hf(x)$ is negative (positive) semi-definite.*

Example 3.1. Let $f(x_1, x_2) = x_1^4 + x_1^2 x_2^4$. Then

$$Df(x) = \begin{bmatrix} 4x_1^3 + 2x_1x_2^4 & 4x_1^2x_2^3 \end{bmatrix}, \quad \text{and} \quad Hf(x) = \begin{bmatrix} 12x_1^2 + 2x_2^4 & 8x_1x_2^3 \\ 8x_1x_2^3 & 12x_1^2x_2^2 \end{bmatrix}.$$

The determinants of the principal submatrices of Hf are $\det(Hf_{11}) = 12x_1^2 + 2x_2^4 \geq 0$ and

$$\det(Hf) = (12x_1^2 + 2x_2^4)(12x_1^2x_2^2) - (8x_1x_2^3)^2 = x_1^2x_2^2(144x_1^2 - 40x_2^4) \gtreqless 0,$$

so that Hf is neither positive nor negative semi-definite. Therefore, f is neither concave nor convex.

Example 3.2 (Cobb-Douglas)**.** Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be given by[3]

$$f(x) = cx_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n},$$

where $c, \alpha_1, \ldots, \alpha_n > 0$. When is f strict concave?

[3]In economics this type of function is called a Cobb-Douglas function. We will encounter it more often later on.

Note that

$$Df(x) = \begin{bmatrix} \frac{\alpha_1}{x_1} f(x) & \cdots & \frac{\alpha_n}{x_n} f(x) \end{bmatrix}, \quad \text{and}$$

$$Hf(x) = \begin{bmatrix} \frac{\alpha_1(\alpha_1-1)}{x_1^2} f(x) & \cdots & \frac{\alpha_1\alpha_n}{x_1 x_n} f(x) \\ \vdots & \ddots & \vdots \\ \frac{\alpha_n\alpha_1}{x_n x_1} f(x) & \cdots & \frac{\alpha_n(\alpha_n-1)}{x_n^2} f(x) \end{bmatrix}.$$

We want to apply Property 1.2 and note that

$$\begin{aligned} \det(-Hf_{kk}) \equiv |-Hf_{kk}| &= (-1)^k f(x)^k \begin{vmatrix} \frac{\alpha_1(\alpha_1-1)}{x_1^2} & \cdots & \frac{\alpha_1\alpha_k}{x_1 x_k} \\ \vdots & \ddots & \vdots \\ \frac{\alpha_k\alpha_1}{x_k x_1} & \cdots & \frac{\alpha_k(\alpha_k-1)}{x_k^2} \end{vmatrix} \\ &= (-1)^k f(x)^k \left(\frac{\alpha_1}{x_1} \cdots \frac{\alpha_k}{x_k} \right) \begin{vmatrix} \frac{\alpha_1-1}{x_1} & \cdots & \frac{\alpha_k}{x_k} \\ \vdots & \ddots & \vdots \\ \frac{\alpha_1}{x_1} & \cdots & \frac{\alpha_k-1}{x_k} \end{vmatrix} \\ &= (-1)^k f(x)^k \left(\frac{\alpha_1}{x_1} \cdots \frac{\alpha_k}{x_k} \right) \begin{vmatrix} \frac{\alpha_1-1}{x_1} & \cdots & \frac{\alpha_1}{x_1} \\ \vdots & \ddots & \vdots \\ \frac{\alpha_k}{x_k} & \cdots & \frac{\alpha_k-1}{x_k} \end{vmatrix} \\ &= (-1)^k f(x)^k \left(\frac{\alpha_1}{x_1^2} \cdots \frac{\alpha_k}{x_k^2} \right) \begin{vmatrix} \alpha_1 - 1 & \cdots & \alpha_1 \\ \vdots & \ddots & \vdots \\ \alpha_k & \cdots & \alpha_k - 1 \end{vmatrix}. \end{aligned}$$

By using several row operations we can compute this determinant as follows.

$$\begin{vmatrix} \alpha_1 - 1 & \cdots & \alpha_1 \\ \vdots & \ddots & \vdots \\ \alpha_k & \cdots & \alpha_k - 1 \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^k \alpha_i - 1 & \sum_{i=1}^k \alpha_i - 1 & \cdots & \sum_{i=1}^k \alpha_i - 1 \\ \alpha_2 & \alpha_2 - 1 & \cdots & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_k & \alpha_k & \cdots & \alpha_k - 1 \end{vmatrix}$$

$$= \left(\sum_{i=1}^k \alpha_i - 1 \right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_2 & \alpha_2 - 1 & \cdots & \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_k & \alpha_k & \cdots & \alpha_k - 1 \end{vmatrix}$$

$$= \left(\sum_{i=1}^k \alpha_i - 1 \right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{vmatrix}$$

$$= \left(\sum_{i=1}^k \alpha_i - 1 \right) \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \end{vmatrix} = \left(\sum_{i=1}^k \alpha_i - 1 \right) (-1)^{k-1}.$$

Therefore,

$$\det(-Hf_{kk}) = (-1)^{2k-1} \left(\frac{\alpha_1}{x_1^2} \cdots \frac{\alpha_k}{x_k^2} \right) f(x)^k \left(\sum_{i=1}^k \alpha_i - 1 \right),$$

which is positive for all $k = 1, \ldots, n$, only if $\sum_{i=1}^n \alpha_i < 1$.

4 Optimization of a Function on $\mathbb{R}^n$

We now turn to optimization problems with functions $f : \mathbb{R}^n \to \mathbb{R}$.

Problem 2. Let $f : A \to \mathbb{R}$, with $A \subseteq \mathbb{R}^n$. Determine the location and type of optima of f on A.

First, like with functions on $\mathbb{R}$, we give a criterion that reduces the number of candidate optimum locations.

Theorem 4.1 (first-order condition)**.** *Let c be an interior point of A. If c is an optimum location for f and the partial derivatives of f exist at c, then $Df(c) = 0$.*

Proof. Let f be differentiable at c and let $u \in \mathbb{R}^n$ be such that $\|u\| = 1$. Consider the function F on some open interval $I = (-\delta, \delta)$, defined by

$$F(t) = f(c + t \cdot u).$$

Since $f(c)$ is an optimum of f, it holds that $F(0)$ is an optimum of F.Since F is differentiable at 0, with $F(0) = Df(c)u$ (chain rule) it follows from Theorem 2.1 that $F'(0) = 0$, i.e. that $Df(c)u = 0$.

The vector u has been chosen arbitrarily, so that $Df(c)u = 0$ for all u. Take $u = e_i$, $i = 1, \ldots, n$, then

$$D_1 f(c) = 0, D_2 f(c) = 0, \ldots, D_n f(c) = 0,$$

i.e. $Df(c) = 0$. ■

So, completely analogous to Section 2 we note that the only possible candidates for an optimum are those interior points c where f is differentiable and $Df(c) = 0$, the boundary points of A, and the points where f is not differentiable.

Definition 4.1 (stationary points)**.** The solutions to the system of equations $Df(x) = 0$ are called the stationary points of f.

This definition is consistent with the definition we gave for the one-dimensional case. There are stationary points that do not lead to a maximum or a minimum.

Definition 4.2 (saddle point)**.** A saddle point of f is a stationary point that is not an optimum location.

So, we get the following algorithm.

Algorithm 2 (finding candidate optima)**.**

1. *Determine the stationary points of f;*
2. *Determine the points where f is not differentiable;*
3. *Determine the boundary points of A.*

We have the following generalization of Theorem 2.3.

Theorem 4.2 (second-order condition)**.** *Let f be C^2 on on A and let $c \in A$ be an interior point of A.*

1. *If $Df(c) = 0$ and $Hf(c)$ is negative (positive) definite, then $f(c)$ is a strict local maximum (minimum) of f.*
2. *If $Df(c) = 0$ and $Hf(c)$ is neither negative nor positive definite, then $f(c)$ is a saddle point of f.*

Proof.

1. We give a sketch of the proof of this part. Suppose $Hf(c)$ is negative definite. Since the second order partial derivatives are continuous, there exists a neighbourhood $U_\varepsilon(c)$, such that $Hf(x)$ is negative semi definite for all $x \in U_\varepsilon(c)$. Take $x \in U_\varepsilon(c)$, $x \neq c$. Then there exists ξ between x and c such that (Taylor)
$$f(x) = f(c) + \frac{1}{2}(x-c)^\top Hf(\xi)(x-c).$$
Since $\xi \in U_\varepsilon(c)$, it holds that $Hf(\xi)$ is positive definite. Therefore
$$\frac{1}{2}(x-c)^\top Hf(\xi)(x-c) \leq 0.$$
Hence, $f(x) \leq f(c)$, for all $x \in U_\varepsilon(c)$. Therefore, $f(c)$ is a local maximum.

2. Follows from $-f(c)$ being a local maximum.

3. Suppose $Hf(c)$ is neither negative nor positive definite (i.e. indefinite). Suppose that $f(c)$ is not a saddle point. Then $f(c)$ is a local optimum. Assume, without loss of generality (wlog) that $f(c)$ is a local minimum of f.

   Let $u \in \mathbb{R}^n$. Since c is a stationary point of f, and hence an interior point of A, there exists $\delta > 0$ such that $c + t \cdot u \in A$ for all $t \in I = (\delta, \delta)$. On I, define the function F by $F(t) = f(c + t \cdot u)$. Then $F(0) = f(c)$ is a local minimum of F. From the chain rule it follows that F is twice differentiable and that
$$F'(t) = Df(c + t \cdot u)u, \quad \text{and} \quad F''(t) = u^\top Hf(c + t \cdot u)u,$$
   for all $t \in I$. Since $F(c)$ is a local minimum of F it holds that $F''(c) \geq 0$. But since u has been chosen arbitrarily this means that $Hf(c)$ is positive semi definite, which is a contradiction.

■

Theorem 4.2 almost always tells us what kind of point a stationary point of f is. The only case where we need further investigations is when $Hf(c)$ is singular and semi-definite (positive of negative). In that case, the Hessian does not give enough information.

Example 4.1. Consider the function $f : \mathbb{R}^4 \to \mathbb{R}$ defined by
$$f(x) = 20x_2 + 48x_3 + 6x_4 + 8x_1x_2 - 4x_1^2 - 12x_3^2 - x_4^2 - 4x_2^3.$$
Then
$$Df(x)^\top = \nabla f(x) = \begin{bmatrix} 8x_2 - 8x_1 \\ 20 + 8x_1 - 12x_2^2 \\ 48 - 24x_3 \\ 6 - 2x_4 \end{bmatrix}.$$

This implies that the stationary points are $(-1, -1, 2, 3)$ and $(5/3, 5/3, 2, 3)$. Furthermore, the Hessian of f equals

$$Hf(x) = \begin{bmatrix} -8 & 8 & 0 & 0 \\ 8 & -24x_2 & 0 & 0 \\ 0 & 0 & -24 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Note that

$$\begin{aligned} \det(Hf(x)_{11}) &= -8 \\ \det(Hf(x)_{22}) &= 64(3x_2 - 1) \\ \det(Hf(x)_{33}) &= -1536(3x_2 + 1) \\ \det(Hf(x)_{44}) &= 3072(3x_2 + 1). \end{aligned}$$

It then follows from Theorem 4.2 that $f(-1, -1, 2, 3)$ is a saddle point of f, whereas $f(5/3, 5/3, 2, 3)$ is a local maximum. Finally, since f is everywhere differentiable and there are no boundary points, there are no other candidate optima.

Can we say something about the global character of optima? Even more so than in the one-dimensional case this is a complex matter. There are three situations where we can say more.

Theorem 4.3 (Weierstrass)**.** *Let $A \subset \mathbb{R}^n$ be compact and non-empty and let $f : A \to \mathbb{R}$ be continuous. Then f has a global maximum and a global minimum on A.*

The proof of this theorem is a bit long and tedious and, therefore, omitted.

Theorem 4.4. *Let f be a C^1 function on A, where $A \subseteq \mathbb{R}^n$ is convex and let c be an interior point of A. If f is concave then $f(c)$ is a global maximum iff $Df(c) = 0$. If f is convex then $f(c)$ is a minimum iff $Df(c) = 0$.*

Proof. We prove the result only for concave functions. Define the first order Taylor polynomial $g_c : \mathbb{R}^n \to \mathbb{R}$ around c, i.e.

$$g_c(x) = f(c) + Df(c)(x - c).$$

Since f is convex it holds that $f(x) \leq g_c(x)$, for all $x \in A$. Since $Df(c) = 0$, we see that $f(x) \leq f(c)$, for all $x \in A$. Therefore, $f(c)$ is a global maximum. ■

Theorem 4.5. *Let A be a closed, non-empty subset of $\mathbb{R}^n$, and let $f : A \to \mathbb{R}$ be continuous such that $f(x) \to -\infty$ as $x_i \to \pm\infty$ (and $x_i \in A$). Then f has a global maximum on A.*

Proof. Assume wlog that $0 \in A$. Then there exists $r$such that $f(x) \leq f(0)$, for all $x \in A$ such that $\|x\| > r$. The set

$$K = \{x \in A : \|x\| \leq r\},$$

is compact and, therefore, f attains a global maximum on K (Weierstrass) at, say, c. Since $0 \in K$, we find that

$$f(c) \geq f(0) \geq f(x), \quad \forall_x \in S \setminus K,$$

so that f has a global maximum on A. ■

Example 4.2. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$, defined by

$$f(x) = x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3.$$

Then

$$\nabla f(x) = \begin{bmatrix} 2x_1 + 2x_2 + 2x_3 \\ 4x_2 + 2x_1 \\ 6x_3 + 2x_1 \end{bmatrix},$$

implying that $(0, 0, 0)$ is the only stationary point. Also,

$$Hf(x) = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 4 & 0 \\ 2 & 0 & 6 \end{bmatrix},$$

implying that $\det(Hf(x)_{11}) = 2 > 0$, $\det(Hf(x)_{22}) = 4 > 0$, $\det(Hf(x)_{33}) = 8 > 0$. Therefore, $Hf(x)$ is positive definite for all $x \in \mathbb{R}^3$ and, hence, f is strictly convex. From Theorem 4.4 it then follows that $f(0, 0, 0, 0)$ is a global minimum.

5 The Implicit Function Theorem

In economics and business problems we want to optimize some function under constraints. For example, suppose that you are the production manager of a company that produces tables, chairs, etc. To make these items you need wood, nails, and labour. Your task is to find an allocation of wood, nails, and labour, so as to produce a certain amount of tables, while minimizing the costs of doing so. Obviously, you cannot just do anything, as there is a fixed way of how to combine the *inputs* (wood, nails, labour) into *outputs* (tables, chairs, etc.). Economists call these transition processes *technologies*. Each one can be thought of as a cooking book recipe: how do you use the ingredients to produce a nice meal. A technology can be written as a function $g_i(x)$, where x denotes the vector of inputs. The costs of using a vector

x of inputs can also be written as a function, say $f(x)$. In general, your problem is to solve the problem

$$\begin{aligned} &\text{minimize } f(x) \\ &\text{such that } g_1(x) = b_1 \\ &\qquad\qquad\vdots \\ &\qquad\qquad g_m(x) = b_m, \end{aligned} \tag{1}$$

where $b_1, \ldots, b_m$ are the pre-specified levels of output you need to produce.

In the next section we will introduce the mathematics of solving such problems. The solution uses the so-called *implicit function theorem.* This theorem deals with the question under what conditions it is possible we can get, from a system of m equations in n variables ($m < n$), m equivalent equations, such that m variables from the vector $x = (x_1, \ldots, x_n)$, can be written as a function of the other $n - m$ variables. In other words, are there functions $\varphi_1, \ldots, \varphi_m$, such that

$$\begin{aligned} x_{n-m+1} &= \varphi_1(x_1, \ldots, x_{n-m}) \\ &\vdots \\ x_n &= \varphi_m(x_1, \ldots, x_{n-m}), \end{aligned} \tag{2}$$

such that this system is equivalent with (1). If, namely, this is possible we can reduce the problem of finding optima of f under (1) to finding the optima of the mapping

$$(x_1, \ldots, x_{n-m}) \mapsto f(x_1, \ldots, x_{n-m}, \varphi_1(x_1, \ldots, x_{n-m}), \ldots, \varphi_m(x_1, \ldots, x_{n-m})), \tag{3}$$

without any constraints. Then we can apply the results from Section 4 to (3) and we are done.

Unfortunately, life is not so simple as the following example illustrates.

Example 5.1. Let $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$, i.e. the unit circle. It is obvious that we cannot describe the entire circle by using a function of the form $x_1 = \varphi_1(x_2)$, or $x_2 = \varphi_2(x_1)$.

But what if we are more modest? Let's look at the unit circle. We can see that for every point $a \in S$, there exists a neighbourhood $U_\varepsilon(a)$, such that all points in $U_\varepsilon(a)$ *can* be written as a function of the form $x_i = f_i(x_j)$. In other words, we can not find a *global* solution to the problem, but a *local* solution is possible.

We also saw in the previous section that solving optimization problems is, in first instance, an entirely local business. Recall that we looked at *local* convexity and concavity to determine the nature of a candidate optimum. So maybe the local result we found in the example is actually not quite that bad. The question then becomes: can we always find such a neighbourhood $U_\varepsilon(a)$, for each point a in the domain?

Suppose, for now that we have found conditions such that, locally, we can rewrite (1) as (2). Then, by defining

$$x^{(1)} = (x_1, \ldots, x_{n-m}), \quad x^{(2)} = (x_{n-m+1}, \ldots, x_n), \quad \text{and} \quad \varphi = (\varphi_1, \ldots, \varphi_m),$$

then optimizing f under (1) is the same as optimizing $F(x^{(1)} := f(x^{(1)}, \varphi(x^{(1)}))$. The Jacobian matrix of g can be written as

$$Dg = \begin{bmatrix} g_{x_1x_1} & \cdots & g_{x_1x_{n-m}} & g_{x_1x_{n-m+1}} & \cdots & g_{x_1x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ g_{x_mx_1} & \cdots & g_{x_mx_{n-m}} & g_{x_mx_{n-m+1}} & \cdots & g_{x_mx_n} \end{bmatrix} =: \begin{bmatrix} Dg^{(1)} & Dg^{(2)} \end{bmatrix}.$$

We can now find the stationary points of F by solving

$$Df \begin{bmatrix} I \\ D\varphi \end{bmatrix} = 0. \tag{4}$$

Since $g(x^{(1)}, \varphi(x^{(1)})) = b =: (b_1, \ldots, b_m)$, we have $Dg = 0$. Therefore,

$$\begin{bmatrix} Dg^{(1)} & Dg^{(2)} \end{bmatrix} \begin{bmatrix} I \\ D\varphi \end{bmatrix} = 0.$$

If $Dg^{(2)}$ is invertible, it then follows from the latter equation that

$$Dg^{(1)} + Dg^{(2)} D\varphi = 0 \iff D\varphi = (Dg^{(2)})^{-1} Dg^{(1)}.$$

Substituting this back into (4) we find that

$$Df \begin{bmatrix} I \\ (Dg^{(2)})^{-1} Dg^{(1)} \end{bmatrix} = 0.$$

The interesting thing is that we can solve this last system of equations *without any knowledge of* φ. This is an essential result, which makes the task of finding the stationary points of the optimization problem reasonably straightforward.

The main question, of course, is under what conditions we can rewrite (1) locally as (2). The answer is given by the following theorem.

Theorem 5.1 (Implicit function theorem). *Let $g_i : S \to \mathbb{R}$, $i = 1, \ldots, m$, be C^1 functions and let S be an open subset of $\mathbb{R}^n$. Consider the system of m equations, with $m < n$,*

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= b_1 \\ &\vdots \\ g_m(x_1, \ldots, x_n) &= b_m. \end{aligned} \tag{5}$$

Let a be a solution to this system. If the Jacobian matrix Dg has rank m, then there exists a neighbourhood where we can solve the m variables corresponding to the m independent columns of $Dg(a)$, uniquely as continuous (partially) differentiable functions $\varphi_1, \ldots, \varphi_m$ of the remaining $n-m$ variables.

In other words, if (wlog) $a = (c, d) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ is a solution to (5) and the last m columns of $Dg(a)$ are linearly independent, then there exist a neighbourhood U around c in $\mathbb{R}^m$, a neighbourhood V around d in $\mathbb{R}^{n-m}$, and a unique C^1 function $\varphi : U \to V$, such that

$$g(x_1, \ldots, x_{n-m}, \varphi(x_1, \ldots, x_{n-m})) = b,$$

for all $(x_1, \ldots, x_{n-m}) \in U$. Note that the partial derivatives of $\varphi_1, \ldots, \varphi_m$, which are continuous, can be expressed in terms of the partial derivatives of $g_1, \ldots, g_m$, by implicit differentiation of the system (5) with respect to $x_{n-m+1}, \ldots, x_n$. We can see this as follows. Assuming (wlog) that $Dg^{(2)}$ is singular in a, then the IFT says that in a neighbourhood of a it holds that

$$\begin{aligned} x_{n-m+1} &= \varphi_1(x_1, \ldots, x_{n-m}) \\ &\vdots \\ x_n &= \varphi_m(x_1, \ldots, x_{n-m}). \end{aligned}$$

It, therefore, follows that

$$D\varphi(x^{(1)}) = -\left[Dg^{(2)}(x^{(1)}, \varphi(x^{(1)}))\right]^{-1} Dg^{(1)}(x^{(1)}, \varphi(x^{(1)})).$$

Proof. In lecture. ■

Example 5.2. Let

$$\begin{aligned} g_1(x_1, x_2, x_3) &= x_1^2 - x_2 - 3x_3^3 = 0 \\ g_2(x_1, x_2, x_3) &= -2x_1 + 2x_2^3 - x_3 = 0. \end{aligned}$$

Differentiation gives the Jacobian

$$Dg = \begin{bmatrix} 2x_1 & -1 & -9x_3^2 \\ -2 & 6x_2^2 & -1 \end{bmatrix}.$$

Note that this matrix has rank 2, since the determinant of the last 2 columns is non-zero:

$$\begin{vmatrix} -1 & -9x_3^2 \\ 6x_2^2 & -1 \end{vmatrix} = 1 + 54x_2^2 x_3^2 \neq 0.$$

From the IFT we now see that for every point a, with $g(a) = 0$, we can, locally, write

$$x_2 = \varphi_1(x_1), \quad \text{and} \quad x_2 = \varphi_2(x_1).$$

We can now apply implicit differentiation and find

$$D\varphi(a) = -\begin{bmatrix} -1 & -9a_3^2 \\ 6a_2^2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2a_1 \\ -2 \end{bmatrix} = -\frac{1}{1+54a_2^2a_3^2} \begin{bmatrix} -1 & 9a_3^2 \\ -6a_2^2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2a_1 \\ -2 \end{bmatrix}.$$

Example 5.3. Let

$$g(x_1, x_2, x_3) = \frac{1}{12}x_1^2 + \frac{1}{3}x_2^2 + \frac{1}{27}x_3^2.$$

Differentiation gives:

$$Dg = \begin{bmatrix} \frac{2x_1}{12} & \frac{2x_2}{3} & \frac{2x_1}{27} \end{bmatrix}.$$

So, $rank(Dg) = 1$, for all $x \neq 0$. Since $g(x) \neq 1$, we conclude from the IFT that around every solution to $g(x) = 1$, one variable can locally be written as a function of the two other variables. Let's take $(0, \sqrt{3}, 0)$. The IFT tells us there is a neighbourhood around $(0, \sqrt{3}, 0)$, where x_2 can be written as a function of x_1 and x_3, i.e. $x_2 = \varphi(x_1, x_3)$. For the derivative of φ in $(0, 0)$, we find that

$$\begin{aligned} D\varphi(0,0) &= -\left(\frac{2x_2}{3}\right)^{-1} \begin{bmatrix} \frac{2x_1}{12} & \frac{2x_3}{27} \end{bmatrix} \Big|_{(x_1,x_2,x_3)=(0,\sqrt{3},0)} \\ &= -\tfrac{1}{2}\sqrt{3} \begin{bmatrix} 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}. \end{aligned}$$

In general there are several ways to write this system of equations as a function of two variables. Consider, for example, the point $(-2, 1, -3)$. Here we could, locally, write $x_1 = \varphi_1(x_2, x_3)$, $x_2 = \varphi_2(x_1, x_3)$, or $x_3 = \varphi_3(x_1, x_2)$, for certain φ_1, φ_2, and φ_3. It will turn out later that the precise choice is irrelevant for solving optimization problems (although a clever choice can make computations easier).

A final reminder: the IFT applies locally! Even if the Jacobian has full row-rank for every solution x with the same set of linearly independent columns of Dg, this does not imply that the function φ is the same.

6 Optimization of a Function on $\mathbb{R}^n$ with Equality Constraints

In this section we consider the following problem.

Problem 3. Let $f : S \to \mathbb{R}$ be differentiable, with $S \subseteq \mathbb{R}^n$ open. Find the location and type of the optima of f on S, given the constraint that $x \in S$ satisfies $g_i(x) = b_i$, $i = 1, \ldots, m$, where all g_i are C^1. We assume that $m < n$.

let us first look at some classical examples from economics and operations research before delving into the mathematics.

Example 6.1 (Utility maximization)**.** Consider a consumer who has to choose a "bundle" of goods from a set $T \subseteq \mathbb{R}^n$.[4] Economists assume that this consumer tries to maximize her "utility" (a fancy word for "well-being"), which is given by some function $U : T \to \mathbb{R}$, called the *utility function.*[5] Obviously, the consumer can only buy bundles that she can afford, i.e. $x \in T$ should be such that it satisfies the *budget constraint* $p^\top x = b$, where p denotes the vector of prices for each good, and b denotes the consumer's budget. So, the consumer's problem is to solve the problem

$$\begin{aligned} &\text{maximize } U : T \to \mathbb{R} \\ &\text{such that } p^\top x = b. \end{aligned}$$

Example 6.2 (Inventory management)**.** A firm has to determine how many times a year (denoted by the variable x_2) a certain quantity of a good (denoted by x_1) has to be taken into its inventory to satisfy a certain demand d. It is assumed that between two orders, the inventory decreases with a constant speed until nothing is left. The average inventory over, say, a year is then $\frac{1}{2}x_1$, with unit cost c_1. Placing an order incurs costs c_2. The firm's choice is now determined by minimizing the total costs under the condition that demand is satisfied:

$$\begin{aligned} &\text{minimize } \tfrac{1}{2}c_1x_1 + c_2x_2 \\ &\text{such that } x_1x_2 = d. \end{aligned}$$

Remark 6.1. *In both examples it is, of course, implicitly assumed that $x \geq 0$. We will discuss this constraint in more detail in the next Section.*

To solve the basic problem of this section we will, in fact, reduce it to a problem of optimization without constraints. We can then use Theorem 4.1 to determine the stationary points of that reduced problem. This reduction can be done in two ways. Firstly, it can, sometimes, be achieved via direct substitution of the constraints in the *objective function* (i.e. the function which is to be optimized). If this method fails, a more general method can be found, which uses an indirect substitution of the constraints in the objective function via the implicit function theorem (Theorem 5.1).

[4] This could be yourself walking through the supermarket trying to buy your groceries. A bundle of goods is then nothing more than a list of the goods you have put in your basket.

[5] later in the course we will build an axiomatic theory of preferences leading to such utility functions.

6.1 The substitution method

The basic idea of the substitution method is to solve the constraints explicitly. If this can be done, we can immediately substitute the m conditions in the objective function, which then becomes a function of $n-m$ variables. The analysis of Section 4 can then immediately be applied. The advantage of this method is that is quick and intuitively appealing. The disadvantage is that is does not always work. The method will be illustrated with two examples.

Example 6.3. Consider the problem

$$\begin{aligned} \text{optimize } f(x_1, x_2, x_3) &:= x_1^2 + x_2^2 + x_3^2 \\ \text{such that } g_1(x_1, x_2, x_3) &:= x_1 + 2x_2 + x_3 = 1 \\ g_2(x_1, x_2, x_3) &:= 2x_1 - x_2 - 3x_3 = 4. \end{aligned}$$

From the constraints we can, for example, solve x_1 and x_2 explicitly as functions of x_3, since it holds that

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 - x_3 \\ 4 + 3x_3 \end{bmatrix},$$

which implies that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 - x_3 \\ 4 + 3x_3 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 - x_3 \\ 4 + 3x_3 \end{bmatrix} = \begin{bmatrix} \frac{9}{5} + x_3 \\ -\frac{2}{5} - x_3 \end{bmatrix}.$$

Substitution in the objective function then gives the reduced optimization problem

$$\text{optimize } F(x_3) := (9/5 + x_3)^2 - (2/5 + x_3)^2 + x_3^2.$$

The stationary points are found from

$$F'(x_3) = 0 \iff 2(9/5 + x_3) - 2(2/5 + x_3) + 2x_3 = 0 \iff x_3 = -7/10.$$

Furthermore, $F''(x_3) = 2 > 0$. Therefore, $f(11/10, 3/10, -7/10)$ is a global strict minimum.

Example 6.4. Consider the problem

$$\begin{aligned} \text{optimize } f(x_1, x_2, x_3) &:= x_1 x_2 \\ \text{such that } g_1(x_1, x_2, x_3) &:= x_1 = 12 - x_2^2 \\ g_2(x_1, x_2, x_3) &:= x_2^2 + x_3^2 = 1. \end{aligned}$$

From g_2 it follows that $|x_2| = \sqrt{1 - x_3^2} \leq 1$ and $x_3^2 = 1 - x_2^2$. Substitution of g_1 in the objective function gives

$$F(x_2) := (12 - x_2^2)x_2.$$

So, the optimization problem is equivalent with the following one:

$$\begin{aligned} &\text{optimize } F(x_2) := (12 - x_2^2)x_2 \\ &\text{such that } |x_2| \leq 1. \end{aligned}$$

The stationary points are given by

$$F'(x_2) = 0 \iff -2x_2^2 + 12 - x_2^2 = 0 \iff x_2^2 - 4 = 0 \iff x_2 = 2 \vee x_2 = -2.$$

These do not satisfy the condition and, therefore, F does not have stationary points. What about the boundary points x such that $|x| = 1$? Note that $F'(x_2) > 0$, for all $x \in [-1, 1]$. Therefore, F attains a strict minimum at $x = -1$ and a strict maximum at $x = 1$. To conclude, $f(11, -1, 0)$ is a strict minimum and $f(11, 1, 0)$ is a strict maximum.

6.2 The method of Lagrange

If we cannot explicitly solve the constraints, it is still – under a mild regularity condition – possible to solve the constraints (locally) implicitly via the implicit function theorem. That is, locally it holds that $x^{(2)} = \varphi(x^{(1)})$ for some unknown function φ. By substituting this function in the objective function we get a reduced objective function F that only depends on $x^{(1)}$. By using the result that $D\varphi = -(Dg^{(2)})^{-1}Dg^{(1)}$, we can compute $DF(x^{(1)})$, without knowing φ:

$$DF(x^{(1)}) = Df \begin{bmatrix} I \\ D\varphi \end{bmatrix} = \begin{bmatrix} I \\ -(Dg^{(2)})^{-1}Dg^{(1)} \end{bmatrix}.$$

We will make this procedure formal in the following theorem.

Theorem 6.1 (Lagrange)**.** *Let $f, g_1, \ldots, g_m : T \to \mathbb{R}$ be C^1 functions on an open set $T \subseteq \mathbb{R}^n$, with $m < n$. Let a be a solution of system (5). If a is an optimum-location for f under the constraint $g(x) = b$, and if the Jacobian matrix $Dg(a)$ has rank m, then there exists a unique vector $\lambda \in \mathbb{R}^m$, such that*

$$Df(a) - \lambda^\top Dg(a) = Df(a) - \sum_{i=1}^{m} \lambda_i \cdot Dg_i(a) = 0^\top. \tag{6}$$

Proof. Assume (wlog) that $x^{(1)} = (x_1, \ldots, x_{n-m})$ and $x^{(2)} = (x_{n-m+1}, \ldots, x_n)$. From Theorem 5.1 it follows that there exists a neighbourhood $U_\varepsilon(a)$ of $\mathbb{R}^n$, such that for all $x \in U_\varepsilon(a)$ it holds that

$$x^{(2)} = \varphi(x^{(1)}),$$

for some function $\varphi : \mathbb{R}^{n-m} \to \mathbb{R}^m$. Substitution in the objective function leads to a new objective function on $U_\varepsilon(a)$:

$$F(x^{(1)}) := f(x^{(1)}, \varphi(x^{(1)})).$$

After defining

$$Df^{(1)} := \begin{bmatrix} f_{x_1} & \cdots & f_{x_{n-m}} \end{bmatrix}, \quad \text{and} \quad Df^{(2)} := \begin{bmatrix} f_{x_{n-m+1}} & \cdots & f_{x_n} \end{bmatrix},$$

applying Theorems 4.1 and 5.1 gives

$$\begin{aligned} DF(a) &= Df^{(1)}(a) + Df^{(2)}(a) D\varphi(a) \\ &= Df^{(1)}(a) - Df^{(2)}(a) \cdot (Dg^{(2)})^{-1} \cdot Dg^{(1)}(a) \\ &= 0^\top. \end{aligned}$$

Define $\lambda \in \mathbb{R}^m$ as

$$\lambda^\top = Df^{(2)}(a) \cdot (Dg^{(2)})^{-1},$$

which turns the first-order condition into

$$DF(a) = Df^{(1)}(a) - \lambda^\top Dg^{(1)}(a).$$

Furthermore, by post-multiplying the definition of λ by $Dg^{(2)}(a)$, and rearranging we find that

$$Df^{(2)}(a) - \lambda^\top Dg^{(2)}(a) = 0^\top.$$

The equation (6) now follows form observing that

$$Dg(a) = \begin{bmatrix} Dg^{(1)}(a) & Dg^{(2)}(a) \end{bmatrix}.$$

■

Remark 6.2. *If $n = 2$ and $m = 1$, (6) becomes*

$$\begin{bmatrix} f_{x_1}(a) & f_{x_2} \end{bmatrix} - \lambda \begin{bmatrix} g_{x_1}(a) & g_{x_2}(a) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

or

$$\begin{aligned} f_{x_1}(a) - \lambda g_{x_1}(a) = 0 \\ f_{x_2}(a) - \lambda g_{x_2}(a) = 0. \end{aligned}$$

Note that if we define the function $\mathscr{L} : \mathbb{R}^{n+m} \to \mathbb{R}$, as

$$\mathscr{L}(x, \zeta) := f(x) - \sum_{i=1}^{m} \zeta_i \left(g_i(x) - b_i \right),$$

the equations (6) and (5) are precisely the necessary conditions for (a, λ) to be a stationary point of $\mathscr{L}$.

A straightforward way of obtaining the necessary conditions (6) and (5) is now to introduce the function $\mathscr{L}$ belonging to the problem at hand and to find the stationary points of $\mathscr{L}$. This leads to the system of equations

$$\begin{aligned} Df(x) - \zeta^\top \cdot Dg(x) &= 0^\top \\ g(x) &= b. \end{aligned} \tag{7}$$

Since the function $\mathscr{L}$ plays such an important role, it gets a special name.

Definition 6.1 (Lagrangian)**.** The Lagrangian belonging to Problem 3 is the function $\mathscr{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\mathscr{L}(x, \xi) = f(x) - \sum_{i=1}^{m} \zeta_i \left(g_i(x) - b_i\right).$$

The ξ_i in Definition 6.1 are called the *Lagrange multipliers*. In order to find the optimum locations for Problem 3 we get the following algorithm.

Algorithm 3. *1. Determine the solutions to system (7).*

2. Determine all x that satisfy (5) and for which $|Dg(x)| < m$.

The next problem, of course, is to find the optima from all these potential locations and to determine what kind of optima they represent. For the points that are found via Theorem 6.1 we have the following results.

Theorem 6.2. *Let $f, g_1, \ldots, g_m : T \to \mathbb{R}$ be C^2 functions, with $T \subseteq \mathbb{R}^n$ open. If (a, λ) satisfies (7), $|Dg(a)| = m$, and*

$$\begin{aligned} &B^\top AB \quad \textit{is negative (positive) definite, where} \\ &A := H\mathscr{L}(a), \quad \textit{and} \\ &B := \begin{bmatrix} I_{n-m} \\ -(Dg^{(2)}(a))^{-1} Dg^{(1)}(a) \end{bmatrix}, \end{aligned}$$

then $f(a)$ is a strict maximum (minimum) of f under the condition $g(x) = b$.

Proof. We give a proof for the case $n = 2$ and $m = 1$. The general proof is analogous, but involves tedious notation. It follows, with the notation from the proof of Theorem 6.1 that, with $F(x_1) = f(x_1, \varphi(x_1))$:

$$DF(a_1) = f_{x_1}(a_1, \varphi(a_1)) + f_{x_2}(a_1, \varphi(a_1)) \cdot \varphi(a_1) = 0.$$

From Theorem 4.2 we know that $f(a)$ is a strict maximum (minimum) if $Hf(a)$ is negative (positive) definite. Application of the chain rule gives that

$$HF(a_1) = f_{x_1x_1} + f_{x_1x_2}\varphi' + f_{x_1x_1}\varphi' + f_{x_2x_2}(\varphi')^2 + f_{x_2}\varphi''.$$

Since $g(x_1, \varphi(x_1)) = b$, it holds that $Dg = 0$. In other words,

$$g_{x_1} + g_{x_2}\varphi' = 0.$$

Differentiating this expression wrt x_1 we see that

$$\begin{aligned} &g_{x_1x_1} + g_{x_1x_2}\varphi' + g_{x_2x_1}\varphi' + g_{x_2x_2}(\varphi')^2 + g_{x_2}\varphi'' = 0 \\ \iff &\varphi'' = -g_{x_2}^{-1} \begin{bmatrix} 1 & \varphi' \end{bmatrix} \begin{bmatrix} g_{x_1x_1} & g_{x_1x_2} \\ g_{x_2x_1} & g_{x_2x_2} \end{bmatrix} \begin{bmatrix} 1 \\ \varphi' \end{bmatrix}. \end{aligned}$$

Substituting this back in HF gives

$$\begin{aligned} HF = &\begin{bmatrix} 1 & \varphi' \end{bmatrix} \begin{bmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{bmatrix} \begin{bmatrix} 1 \\ \varphi' \end{bmatrix} - f_{x_2} \begin{bmatrix} 1 & \varphi' \end{bmatrix} \begin{bmatrix} g_{x_1x_1} & g_{x_1x_2} \\ g_{x_2x_1} & g_{x_2x_2} \end{bmatrix} \begin{bmatrix} 1 \\ \varphi' \end{bmatrix} \\ = &B^\top H\mathscr{L}B, \end{aligned}$$

from which the result follows by applying Theorem 4.2. ■

Remark 6.3. *This condition can also be interpreted as follows: the matrix $H\mathscr{L}(a)$ must be negative (positive) definite on the null-space of the matrix $Dg(a)$.*

For boundary points, points where f is non-differentiable, or where $|Dg(a)| < m$, we can, in general, not say much more then that further investigation is needed.

Example 6.5. Consider the problem

$$\begin{aligned} \text{optimize } f(x_1, x_2) &:= x_1^2 + x_2^2, \quad x_1, x_2 \in \mathbb{R} \\ \text{such that } g(x_1, x_2) &:= x_1^2 + x_1x_2 + x_2^2 = 3. \end{aligned}$$

The Lagrangian of this problem equals

$$\mathscr{L}(x_1, x_2, \xi) = x_1^2 + x_2^2 - \zeta(x_1^2 + x_1x_2 + x_2^2 - 3),$$

which gives first-order conditions

$$\begin{aligned} \frac{\mathscr{L}}{x_1} &= 2x_1 - \lambda(2x_1 + x_2) = 0 \\ \frac{\mathscr{L}}{x_2} &= 2x_2 - \lambda(x_1 + 2x_2) = 0 \\ \frac{\mathscr{L}}{\zeta} &= x_1^2 + x_1x_2 + x_2^2 - 3 = 0, \end{aligned}$$

which gives the stationary points

$$\begin{gathered}(1,1) \text{ and } (-1,-1), \text{ with } \lambda = 2/3 \text{ and} \\ (\sqrt{3},-\sqrt{3}) \text{ and } (-\sqrt{3},\sqrt{3}), \text{ with } \lambda = 2.\end{gathered}$$

Note that

$$Dg(x) = \begin{bmatrix} 2x_1 + x_2 & 2x_2 + x_1 \end{bmatrix},$$

so that $|Dg(x)| = 1$, for all $(x_1, x_2) \neq (0,0)$. Since $g(0,0) \neq 3$, we can say that $|Dg(x)| = 1$, for all (x_1, x_2) that satisfy the condition. The stationary points above are, therefore, the only possible optimum locations. We now find that

$$B^\top AB = \begin{bmatrix} x_1 + 2x_2 & -2x_1 - x_2 \end{bmatrix} \begin{bmatrix} 2-2\lambda & -\lambda \\ -\lambda & 2-2\lambda \end{bmatrix} \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 - x_2 \end{bmatrix} \cdot \frac{1}{(x_1 + 2x_2)^2}.$$

In the stationary points $(\sqrt{3},-\sqrt{3})$ and $(-\sqrt{3},\sqrt{3})$ with $\lambda = 2$, this quadratic form equals $B^\top AB = -8 < 0$, implying that $f(\sqrt{3},-\sqrt{3})$ and $f(-\sqrt{3},\sqrt{3})$ are strict local maxima. Furthermore, for the stationary points $(1,1)$ and $(-1,-1)$ with $\lambda = 2/3$, this quadratic form equals $B^\top AB = 8 > 0$, implying that $f(1,1)$ and $f(-1,-1)$ are strict local minima.

Example 6.6. Consider the problem

$$\begin{aligned} \text{optimize } & f(x_1,x_2,x_3) := x_1 + x_2 + x_3, \quad x_1,x_2,x_3 \in \mathbb{R} \\ \text{such that } & g_1(x_1,x_2,x_3) := x_1 - x_2 - x_3 = 0 \\ & g_2(x_1,x_2,x_3) := x_1^2 + x_2^2 + x_3^2 = 6. \end{aligned}$$

First note that

$$Dg(x) = \begin{bmatrix} 1 & -1 & -1 \\ 2x_1 & 2x_2 & 2x_3 \end{bmatrix}.$$

Hence,

$$|Dg(x)| = \begin{vmatrix} 1 & -1 & -1 \\ 0 & 2(x_1 + x_2) & 2(x_1 + x_3) \end{vmatrix}.$$

From this we see that $|Dg(x)| < 2$ iff $x_2 = -x_1$ and $x_3 = -x_1$. This, however, leads to a system of equations g_1, g_2 with no solutions. So, for all (x_1, x_2, x_3) such that $g_1(x_1,x_2,x_3) = 0$ and $g_2(x_1,x_2,x_3) = 6$, it holds that $|Dg(x)| = 2$. In other words, we can use the Lagrangian, which is given by

$$\mathscr{L}(x_1,x_2,x_3,\zeta_1,\zeta_2) = x_1 + x_2 + x_3 - \zeta_1(x_1 - x_2 - x_3) - \zeta_2(x_1^2 + x_2^2 + x_3^2 - 6).$$

The necessary conditions for an optimum are, therefore,

$$\begin{aligned}
&\text{(i)} && 1-\lambda_1-2\lambda_2 a_1=0\\
&\text{(ii)} && 1+\lambda_1-2\lambda_2 a_2=0\\
&\text{(iii)} && 1+\lambda_1-2\lambda_2 x_3=0\\
&\text{(iv)} && x_1-x_2-x_3=0\\
&\text{(v)} && x_1^2+x_2^2+x_3^2=6
\end{aligned}$$

From (ii) and (iii) we find $\lambda_2(x_2-x_3)$, so that $\lambda_2=0 \vee x_2=x_3$. Consider two cases.

1. $\lambda_2=0$.
Substitution in (i), (ii), and (iii) gives a system with no solutions.

2. $x_2=x_3$.
From (iv) it follows that $x_1=2x_2$, so that (v) gives that $x_2^2=1$. In other words, we find the stationary points $(2,1,1)$, with $\lambda_2=1/3$, and $(-2,-1,-1)$, with $\lambda_2=-1/3$.

3. Let $a=(2,1,1)$. Then

$$\begin{aligned}
Dg(a) &= \begin{bmatrix} Dg^{(2)} & Dg^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 4 & 2 & 2 \end{bmatrix}\\
\Rightarrow -\left(Dg^{(2)}(a)\right)^{-1}Dg^{(1)} &= -\begin{bmatrix} 1 & -1 \\ 4 & 2 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}\\
\Rightarrow B^\top AB &= \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2\zeta_2 & 0 & 0 \\ 0 & -2\zeta_2 & 0 \\ 0 & 0 & -2\zeta_2 \end{bmatrix}_{\zeta_2=1/3}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = -\frac{4}{3} < 0.
\end{aligned}$$

So, $f(2,1,1)$ is a strict local maximum.

4. Let $a=(-2,-1,-1)$. Then

$$\begin{aligned}
Dg(a) &= \begin{bmatrix} Dg^{(2)} & Dg^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -4 & -2 & -2 \end{bmatrix}\\
\Rightarrow -\left(Dg^{(2)}(a)\right)^{-1}Dg^{(1)} &= -\begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}^{-1}\begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}\\
\Rightarrow B^\top AB &= \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} -2\zeta_2 & 0 & 0 \\ 0 & -2\zeta_2 & 0 \\ 0 & 0 & -2\zeta_2 \end{bmatrix}_{\zeta_2=-1/3}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \frac{4}{3} > 0.
\end{aligned}$$

So, $f(-2,-1,-1)$ is a strict local minimum.

For the investigation of the global character of optima we present two theorems. The first is by applying Weierstrass's theorem. A second criterion is given by the following theorem.

Theorem 6.3. *Let $f, g_1, \ldots, g_m : T \to \mathbb{R}$ be C^1 functions, with $T \subseteq \mathbb{R}^n$ open. If a and λ satisfy (7), and the Lagrangian is concave (convex), then $f(a)$ is a global maximum (minimum) for f under the condition $g(x) = b$.*

Proof. Since a and λ satisfy (7), it holds that a is a stationary point for the Lagrangian $\mathscr{L}$. From Theorem 4.2 it follows that $\mathscr{L}(a)$ is a global maximum (minimum) for $\mathscr{L}$. In particular, it holds that $\mathscr{L}(x) \leq (\geq) \mathscr{L}(a)$, for all x such that $g(x) = b$. Since $g(a) = b$, it follows that $f(x) \leq (\geq) f(a)$, for all x such that $g(x) = b$. ■

Example 6.7. Let us reconsider Example 6.6. From g_2 it follows that $|x_i| \leq \sqrt{6}$, all $i = 1, 2, 3$. So, the area over which optimization takes place is bounded. The point $(2, 1, 1)$ satisfies both g_1 and g_2, so the domain is also non-empty. Finally, the domain is closed (it is an ellipse). According to Weierstrass, f, therefore, has a global maximum and a global minimum on the domain. The only candidates are $f(2, 1, 1)$ and $f(-2, -1, 1)$, respectively.

Finally we say something about the meaning of the Lagrange multipliers ζ_i, $i = 1, \ldots, m$. The optimal value f^* of the objective function can be seen as a function of the values b_i, $i = 1, \ldots, m$. In management applications, for example, these b_i's represent production inputs. The question then is whether the firm should increase the inventory of, say, input j, i.e. should b_j be increased? The boundary case between a "yes" and a "no" answer occurs if the marginal improvement in the optimal value f^* equals the price of an additional unit of the input. This marginal improvement turns out to be equal to the Lagrange multiplier ζ_j. For this reason, the Lagrange multipliers are also called *shadow prices*. They are an indication of the change in f^*, resulting from a marginal change in the b_i's.

Theorem 6.4. *Let $f, g_1, \ldots, g_m : T \to \mathbb{R}$ be C^1 functions, with $T \subseteq \mathbb{R}^n$ open. Let $f(a)$ be an optimum of f under the condition $g(x) = b_0$, with appropriate Lagrange multipliers λ, and let*

$$\begin{vmatrix} H\mathscr{L}(a) & -Dg(a)^\top \\ Dg(a) & 0 \end{vmatrix} \neq 0.$$

Then it holds that, for all $i = 1, \ldots, m$,

$$f^*_{b_i} = \lambda_i.$$

Proof. Consider the following system of $n+m$ equations in $2m+n$ unknowns:

$$\begin{aligned} Df(x)^\top - Dg(x)^\top \zeta &= 0 \\ g(x) - b &= 0. \end{aligned}$$

The point (b_0, a, λ) is a solution to this system, since (a, λ) is a stationary point of $\mathscr{L}$, belonging to the problem with constraint $g(x) = b_0$. Since the matrix

$$\begin{bmatrix} 0 & H\mathscr{L} & -Dg^\top \\ -I & Dg & 0 \end{bmatrix},$$

has rank $n+m$ in (b_0, a, λ), the implicit function theorem tells us that in a neighbourhood $U_\varepsilon(b_0, a, \lambda)$, the variables x and ζ can be written as differentiable functions of b:

$$x = \varphi(b), \quad \text{and} \quad \zeta = \psi(b).$$

If we implicitly differentiate the system $g(x) = b$ with respect to b_i (for fixed i) we then get

$$Dg_j(x)\varphi_{b_i}(b) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}, \quad j = 1, \ldots, m.$$

So, for all $i = 1, \ldots, m$, it holds that

$$\begin{aligned} f^*_{b_i}(b_0) &= Df(a)\varphi_{b_i}(b_0) \\ &= \sum_{j=1}^{m} \lambda_j Dg_j(a)\varphi_{b_i}(b_0) \\ &= \lambda_i. \end{aligned}$$

■

Remark 6.4. *This theorem will also turn out to play a crucial role in the proof of the Kuhn-Tucker theorem in the next section.*

7 Optimization of a Function on $\mathbb{R}^n$ with Inequality Constraints

In this section we consider the following problem.

Problem 4. Let $f : S \to \mathbb{R}$ be differentiable, with $S \subseteq \mathbb{R}^n$ open. Find the location of the maxima of f on S, given the constraint that $x \in S$ satisfies $g_i(x) \leq b_i$, $i = 1, \ldots, m$, where all g_i are C^1. We assume that $m < n$.

Remark 7.1. *The problem above encompasses all types of optimization problems with inequalities. To find the minimum locations of f, we essentially look for maximum locations of $-f$. Inequalities of the type $g(x) \geq 0$ can be transformed into $-g(x) \leq -b$. Finally, equalities of the type $g(x) = b$ can be transformed into $g(x) \leq b$ and $-g(x) \leq -b$. (This last transformation is usually not advisable; see a later remark).*

To solve the above problem we, again, first formulate a theorem that reduces the number of potential points where a maximum can be attained. We need the following terminology. The i-th constraint in Problem 4 is called a *binding constraint* in $a \in \mathbb{R}^n$ if $g_i(a) = b_i$.

Theorem 7.1 (Kuhn-Tucker). *Let $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ be C^1 functions, and let $a \in \mathbb{R}^n$ be a solution to Problem 4. If the gradient vectors of the binding constraints are independent, then there exists a vector $\lambda \in \mathbb{R}^m$, $\lambda \geq 0$, such that*

$$Df(a) - \lambda^\top Dg(a) = 0^\top \tag{8}$$

$$\lambda^\top (g(a) - b) = 0. \tag{9}$$

Proof. Choose $\lambda_i = 0$ if the i-th constraint is not binding. The binding constraints give a maximization problem with equality constraints that satisfies the conditions of Theorem 6.1. For the corresponding λ_i's, choose the Lagrange multipliers of this system. Obviously, this satisfies (8) and (9).

The only thing that remains to be shown is that $\lambda \geq 0$. To establish this we apply Theorem 6.1 to the maximization problem with just the binding constraints. To keep things simple let us assume that all conditions of Theorem 6.3 are satisfied.[6] Suppose that $\lambda_i < 0$. Theorem 6.3 then gives that

$$f^*_{b_i}(b) = \lambda_i < 0,$$

implying that f^* is a strictly decreasing function of b_i in a neighbourhood of b. So, there exists $\varepsilon > 0$, such that

$$f^*(b - \varepsilon e_i) > f^*(b).$$

So, the system with a more strict condition $g(x) \leq b - \varepsilon e_i$, has a higher maximum value f^*. This is a contradiction. ∎

Remark 7.2. *We make two remarks on this theorem.*

[6] This is not needed, but the proof becomes quite tedious without it.

1. *If the conditions of Theorem 7.1 are satisfied, then the following system of equations gives a collection of possible maximum locations.*

$$\begin{aligned} Df(x) - \zeta^\top Dg(x) &= 0^\top \\ g(x) &\leq b \\ \zeta &\geq 0 \\ \zeta^\top (g(x) - b) &= 0 \end{aligned} \tag{10}$$

2. *If, apart from inequality constraints, there are also equality constraints, say*

$$\begin{aligned} g_i(x) &= b_i, \quad \text{for } i = 1, \dots, k \\ g_i(x) &\leq b_i, \quad \text{for } i = k+1, \dots, m \end{aligned}$$

   then we can use a combination of Theorems 6.1 and 7.1. The system to be solved is then determined by (7) for $1 \leq i \leq k$, *and (10) for* $k+1 \leq i \leq m$.

Algorithm 4. *The set of possible maximum locations for Problem 4 is the set containing:*

1. *all solutions to (10),*

2. *all points where at least one constraint is binding en the corresponding gradient vectors are linearly dependent.*

Can we, from this set of points, find the maximum locations? As usual we might be able to use Weierstrass (Theorem 4.3), or use the fact that f is strictly concave (Theorem 4.4). We also have the following result.

Theorem 7.2. *Consider the Lagrangian* $\mathscr{L}(x, \zeta) = f(x) - \zeta^\top (g(x) - b)$ *in a stationary point* $(a, \lambda) \in \mathbb{R}^n \times \mathbb{R}^m$ *of (10). If* (a, λ) *satisfies*

(i) $\mathscr{L}(x, \lambda) \leq \mathscr{L}(a, \lambda)$, *all* x,

(ii) $g(a) \leq b$,

(iii) $\lambda \geq 0$,

(iv) $\lambda^\top (g(a) - b) = 0$,

then $f(a)$ *is a global maximum for Problem 4.*

Remark 7.3. *Condition (iv) is called the* complementary slackness condition.

Proof. Note that

$$
\begin{aligned}
f(x) - f(a) &\underset{(i)}{\leq} \lambda^\top (g(x) - g(a)) \\
&\underset{(iv)}{=} \lambda^\top (g(x) - b) \\
&\underset{(iii)}{\leq} 0, \quad \text{all } x \text{ with } g(x) \leq b.
\end{aligned}
$$

■

Condition (i) implies that a is a maximum location for the Lagrangian $\mathscr{L}$. By applying Theorem 4.2 to $\mathscr{L}$ whether a candidate maximum location satisfies (i). The sufficient conditions in Theorem 7.2 can not always be used. There exist stronger ones, but we will not discuss them here.

Remark 7.4. *For applications of the theory in this section one could interpret the quantities as follows:*

$$
\begin{aligned}
x&: \textit{production levels,} \\
g(x)x&: \textit{used levels of production inputs,} \\
b&: \textit{levels of production inputs,} \\
\lambda&: \textit{shadow prices of inputs,} \\
f(x)&: \textit{total profits.}
\end{aligned}
$$

Finally, three examples of the theory.

Example 7.1. Consider the problem

$$
\begin{aligned}
\text{maximize } & f(x_1, x_2) := -x_1^2 - x_1x_2 - x_2^2 \\
\text{such that } & g_1(x_1, x_2) := x_1 - 2x_2 \leq -1 \\
& g_2(x_1, x_2) := 2x_1 + x_2 \leq 2 \\
& g_3(x_1, x_2) := -x_1 \leq 0 \\
& g_4(x_1, x_2) := -x_2 \leq 0.
\end{aligned}
$$

The Jacobian matrix of g equals

$$
Dg(x) = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

System (10) becomes

$$\begin{cases} -2x_1 - x_2 - \zeta_1 - 2\zeta_2 + \zeta_3 &= 0 \\ -x_1 - 2x_2 + 2\zeta_1 - \zeta_2 + \zeta_4 &= 0 \end{cases} \tag{11}$$

$$\begin{cases} x_1 - x_2 & \leq -1 \\ 2x_1 + x_2 & \leq 2 \\ x_1 & \geq 0 \\ x_2 & \geq 0 \end{cases} \tag{12}$$

$$\zeta_i \geq 0, \quad i = 1, 2, 3, 4 \tag{13}$$

$$\begin{cases} \zeta_1(x_1 - 2x_2 + 1) &= 0 \\ \zeta_2(2x_1 + x_2 - 2) &= 0 \\ \zeta_3(-x_1) &= 0 \\ \zeta_4(-x_2) &= 0. \end{cases} \tag{14}$$

From the first and third inequalities of (12) it follows that $x_2 > 0$. From (14) we then obtain: $\zeta_4 = 0$. From (11), $x_1 \geq 0$, $x_2 > 0$, and $\zeta_2 \geq 0$, it then follows that $\zeta_1 > 0$ and $\zeta_3 > 0$. Complementary slackness then gives

$$\begin{cases} x_1 - 2x_2 + 1 &= 0 \\ -x_1 &= 0 \end{cases} \iff (x_1, x_2) = (0, 1/2).$$

This implies that the second condition is not binding and, hence, that $z_2 = 0$. Plugging all this back into (11) then reveals that $\zeta_1 = 1/2$ and $\zeta_3 = 1$. So, there is only one point (a, λ) that satisfies 10, with

$$a = (0, 1/2), \quad \text{and} \quad \lambda = (1/2, 0, 1, 0).$$

Are there other points that could be maximum locations? First, notice that at most two constraints can be binding. Since every $k \times 2$-submatrix of Dg has full rank ($k = 1, 2$), there are no other candidate locations.

For (a, λ), the corresponding Lagrangian equals

$$\mathscr{L}(x, \lambda) = -x_1^2 - x_1x_2 - x_2^2 - \tfrac{1}{2}x_1 + x_2 - \tfrac{1}{2} + x_1.$$

The Hessian of this function is

$$H\mathscr{L}(x, \lambda) = \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix},$$

which is negative definite. So, a is a maximum location for $\mathscr{L}(x, \lambda)$ (according to Theorem 4.2). So, Theorem 7.2 tells us that $f(a)$ is a maximum.

Example 7.2. An electricity utility with variable capacity k, partitions a day in different periods $1, 2, \ldots, n$ and delivers, in those periods, electricity levels $x_1, \ldots, x_n$, at varying prices $p_1, \ldots, p_n$. Suppose that the quantities $a_1, \ldots, a_n$, and the capacity k_0 have been chosen such that it maximizes profits. If the function $c(\cdot)$ denotes the costs, accruing from levels $x_1, \ldots, x_n$, and the function $d(\cdot)$ denotes the costs of keeping capacity k, then $a_1, \ldots, a_n, k_0$ is a solution to the problem:

$$\begin{aligned}&\text{maximize } p^\top x - c(x) - d(k)\\&\text{such that } x_i \leq k, \quad i \in \{1, 2, \ldots, n\}.\end{aligned}$$

System (10) then becomes

$$p - Dc(x)^\top - \lambda = 0 \tag{15}$$

$$-d'(k) + \sum_{i=1}^{n} \lambda_i = 0 \tag{16}$$

$$x_i \leq k, \quad i = 1, \ldots, n$$

$$\lambda_i \geq 0, \quad i = 1, \ldots, n$$

$$\lambda_i(x_i - k) = 0, \quad i = 1, \ldots, n \tag{17}$$

From (17) it follows that either $x_i = k$, or $\lambda_i = 0$. If $\lambda_i = 0$, it follows form (15) that

$$p_i = c_{x_i}(x).$$

In words: if capacity is not fully used, then the price of electricity equals the marginal costs of delivering it. If $x_i = k$, then (15) yields

$$p_i = c_{x_i}(x) + \lambda_i.$$

In words: if capacity is fully used (peak hours), then the price of electricity equals the marginal costs of delivering it plus an additional premium. Form (16) one can see that the premium is related to the marginal cost of keeping up the capacity k. So, the costs of keeping capacity k are fully borne by customers who buy electricity during the peak hours.

Finally, note that $Dg(x) = I$. This implies that every number of binding constraints has linearly independent gradient vectors. This implies that the only candidate maximum locations are given by the above system.

Example 7.3. Let

$$V = \{(x, x_2) \in \mathbb{R}^2 | x_1^2 + x_2^2 \leq 9, x_1^2 - x_2^2 \leq 1\}.$$

Suppose we want to maximize $x_1^2 x_2$ over V. We can write this as a maximization problem

$$
\begin{aligned}
\text{maximize } & f(x_1, x_2) := x_1^2 x_2 \\
\text{such that } & g_1(x_1, x_2) := x_1^2 + x_2^2 \leq 9 \\
& g_2(x_1, x_2) := x_1^2 - x_2^2 \leq 1.
\end{aligned}
$$

Note that V is non-empty, bounded, and closed. Since f is continuous, Weierstrass tells us that f attains a maximum on V. Consider the Lagrangian

$$\mathscr{L}(x, \zeta) = x_1^2 x_2 - \zeta_1(x_1^2 + x_2^2 - 9) - \zeta_2(x_1^2 - x_2^2 - 1).$$

This leads to system (10):

$$2x_1x_2 - 2\zeta_1 x_1 - 2\zeta_2 x_1 = 0 \tag{18}$$

$$x_1^2 - 2\zeta_1 x_2 + 2\zeta_2 x_2 = 0 \tag{19}$$

$$x_1^2 + x_2^2 \leq 9 \tag{20}$$

$$x_1^2 - x_2^2 \leq 1 \tag{21}$$

$$\zeta_1 \geq 0 \tag{22}$$

$$\zeta_2 \geq 0 \tag{23}$$

$$\zeta_1(x_1^2 + x_2^2 - 9) = 0 \tag{24}$$

$$\zeta_2(x_1^2 - x_2^2 - 1) = 0 \tag{25}$$

Consider the following four cases.

Case 1. Neither of the two constraints is binding. Therefore, $\zeta_1 = \zeta_2 = 0$. According to (19) it then follows that $x_2 = 0$, implying that $f = 0$.

Case 2. The second constraint is binding, the first is non-binding. So, $\zeta_1 = 0$ and $\zeta_2 \neq 0$. From (18) it then follows that $2x_1(x_2 - \zeta_2) = 0$, so that $x_1 = 0$ or $x_2 = \zeta_2$. If $x_1 = 0$, then $f = 0$. If $x_2 = \zeta_2$, then (19) gives that $x_1^2 + 2\zeta_2^2 = 0$, which gives an inconsistent system. Finally, $Dg_2(x) = 2x_1 - 2x_2$, so that $|Dg_2| < 1$ if $x_1 = x_2 = 0$, which does not satisfy g_2. So, step 2 from Algorithm 4 does not lead to extra candidates.

Case 3. The first constraint is binding, the second is non-binding. Therefore, $\zeta_2 = 0$ and $\zeta_1 \neq 0$. From (18) it then follows that $2x_1(x_2 - \zeta_1) = 0$, implying that $x_1 = 0$, or $x_2 = \zeta_1$. If $x_1 = 0$, the $f = 0$. If $x_2 = \zeta_1$, we get from (19) that $x_1^2 = 2\zeta_1^2$ and from (24) that $x_1^2 = 9 - \zeta_1^2$. This implies that $\zeta_1 = \sqrt{3}$, $x_1^2 = 6$, and $x_2 = \sqrt{3}$.

But, these values do not satisfy (21). Finally, $Dg_1(x) = 2x_1 + 2x_2$, so that $|Dg_1| < 1$ if $x_1 = x_2 = 0$, which does not satisfy g_1. So, step 2 from Algorithm 4 does not lead to extra candidates.

Case 4. Both constraints are binding, implying that $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$. From (24) and (25) we then find that $x_1 = \pm\sqrt{5}$ and $x_2 = \pm 2$.

If $x = (\sqrt{5}, 2)$, then (18) and (19) give that $\zeta_1 = 13/8$, $\zeta_2 = 3/8$, and $f = 10$.

If $x = (-\sqrt{5}, 2)$, then (18) and (19) give that $\zeta_1 = 13/8$, $\zeta_2 = 3/8$, and $f = 10$.

If $x = (\sqrt{5}, -2)$, then $f = -10$.

If $x = (-\sqrt{5}, -2)$, then $f = -10$.

Note that

$$Dg(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -2x_2 \end{bmatrix}.$$

So, $|Dg(x)| < 2$ iff $-8x_1x_2 = 0$, i.e. if $x_1 = 0$ or $x_2 = 0$. If $x_1 = 0$, then g_1 gives that $x_2 = \pm 3$. However, $(0, \pm 3)$, does not satisfy g_2, so this does not give additional candidates. Analogous computations reveal the same for $x_2 = 0$.

So, all possible maximum locations are

$$\{(0, x_2), (\sqrt{5}, 2), (\sqrt{5}, -2), (-\sqrt{5}, 2), (-\sqrt{5}, -2)\}.$$

Simple computations give that f attains the global maximum 10, at $(\sqrt{5}, 2)$ and $(-\sqrt{5}, 2)$.