Exercises 1

Mathematical economics

1. Consider the function $f : (0, \infty) \to \mathbb{R}$, with

$$f(x) = \begin{cases} x + 1 & \text{if } 0 < x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Show that this function has a global maximum and a global minimum, but that the conditions of the Weierstrass theorem are not satisfied.

2. Determine the optimum locations of the function $f : [0, 2] \to \mathbb{R}$, where

$$f(x) = \frac{x^2 + 1}{x}.$$

Also investigate their nature.

3. Consider the function $f : [0, 2] \to \mathbb{R}$, with

$$f(x) = \begin{cases} x^2 - x + \frac{1}{4} & \text{if } 0 \leq x < 1 \\ -x^3 + \frac{3}{2}x^2 - \frac{9}{4}x + 3\frac{1}{4} & \text{if } 1 \leq x \leq 2. \end{cases}$$

(a) Are the conditions of the Weierstrass theorem satisfied?

(b) Determine the location and the nature of the optimum points of f.

4. Consider the Cobb-Douglas function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with

$$f(x) = x_1^\alpha x_2^\beta, \quad \alpha, \beta > 0, \quad \alpha + \beta \leq 1.$$

Let $a \in \mathbb{R}^2_+$ and define

$$V = \{x \in \mathbb{R}^2_+ | f(x) \geq f(a)\}.$$

(a) Show that f is concave on $\mathbb{R}^2_+$.

(b) Show that V is a convex set.

5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$, with

$$f(x_1, x_2) = (x_2 - x_1^2)(x_2 - 2x_1^1),$$

and the functions $g_k, h_1, h_2 : \mathbb{R} \to \mathbb{R}$, with

$$\begin{aligned} g_k(x) &= f(x, kx), \quad k \neq 0, \\ h_1(x) &= f(x, 0), \\ h_2(x) &= f(0, x). \end{aligned}$$

Show that the functions g_k, h_1, h_2 have a local minimum in 0, but that $(0, 0)$ is a saddle-point for f.

6. Determine the location and the nature of the optima of the following functions:

   (a) $f : \mathbb{R}^2 \to \mathbb{R}$, with $f(x_1, x_2) = x_1^4 + x_2^4 - 2x_1^2 + 4x_1x_2 - 2x_2^2$,

   (b) $f : \mathbb{R}^3 \to \mathbb{R}$, with $f(x_1, x_2, x_3) = x_1^3 + 3x_1x_2 + 3x_1x_3 + x_2^3 + 3x_2x_3 + x_3^3$.

7. Determine the location and the nature of the optima of the following functions:

   (a) $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, with $f(x_1, x_2) = x_1^2 + 2x_2^2$,

   (b) $f : [-1, 1] \times [-1, 1] \to \mathbb{R}$, with $f(x_1, x_2) = x_1^3 - 3x_2^2$.

8. Consider the function $f : \mathbb{R}^2_+ \to \mathbb{R}$, with

   $$f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1 - 2x_2.$$

   Show that f is convex on $\mathbb{R}^2_+$, and determine the global minimum of f on $\mathbb{R}^2_+$.

9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function, let $S \subset \mathbb{R}^n$ be a convex set, and let a be a boundary point of S. Prove that if a is a global maximum location for f on S, then it holds that

   $$Df(a) \cdot (a - x) \geq 0, \quad \text{for all } x \in S.$$

   (Hint: consider the function $g : [0, 1] \to \mathbb{R}$, with $g(\lambda) = f(a) - f(\lambda x + (1 - \lambda)a)$.)