Topic 7 Self Assessment I
Partial Differentiation.

1. **Find all first and second order partial derivatives of the following functions:**

(i) $y = x^2 + 2xz + z^2$

$\frac{\partial y}{\partial x} = 2x + 2z$ $\qquad \frac{\partial^2 y}{\partial x^2} = 2$ $\qquad \frac{\partial^2 y}{\partial x \partial z} = 2$

$\frac{\partial y}{\partial z} = 2x + 2z$ $\qquad \frac{\partial^2 y}{\partial z^2} = 2$ $\qquad \frac{\partial^2 y}{\partial z \partial x} = 2$

(ii) $y = x^2 z^2 + 3z$

$\frac{\partial y}{\partial x} = 2xz^2$ $\qquad \frac{\partial^2 y}{\partial x^2} = 2z^2$ $\qquad \frac{\partial^2 y}{\partial x \partial z} = 4xz$

$\frac{\partial y}{\partial z} = 2x^2 z + 3$ $\qquad \frac{\partial^2 y}{\partial z^2} = 2x^2$ $\qquad \frac{\partial^2 y}{\partial z \partial x} = 4xz$

(iii) $y = \frac{x}{z} = xz^{-1}$

$\frac{\partial y}{\partial x} = z^{-1} = \frac{1}{z}$ $\qquad \frac{\partial^2 y}{\partial x^2} = 0$ $\qquad \frac{\partial^2 y}{\partial x \partial z} = -z^{-2} = -\frac{1}{z^2}$

$\frac{\partial y}{\partial z} = -xz^{-2} = -\frac{x}{z^2}$ $\qquad \frac{\partial^2 y}{\partial z^2} = 2xz^{-3} = \frac{2x}{z^3}$ $\qquad \frac{\partial^2 y}{\partial z \partial x} = -z^{-2} = -\frac{1}{z^2}$

(iv) $y = 10x^{0.5} z^{0.2}$

$\frac{\partial y}{\partial x} = 5x^{-0.5} z^{0.2}$ $\qquad \frac{\partial^2 y}{\partial x^2} = -2.5x^{-1.5} z^{0.2}$ $\qquad \frac{\partial^2 y}{\partial x \partial z} = x^{-0.5} z^{-0.8}$

$\frac{\partial y}{\partial z} = 2x^{0.5} z^{-0.8}$ $\qquad \frac{\partial^2 y}{\partial z^2} = -1.6x^{0.5} z^{-1.8}$ $\qquad \frac{\partial^2 y}{\partial z \partial x} = x^{-0.5} z^{-0.8}$

(v) $y = 5xz^2 + 5x^2 z$

$\frac{\partial y}{\partial x} = 5z^2 + 10xz$ $\frac{\partial^2 y}{\partial x^2} = 10z$ $\frac{\partial^2 y}{\partial x \partial z} = 10z + 10x$

$\frac{\partial y}{\partial z} = 10xz + 5x^2$ $\frac{\partial^2 y}{\partial z^2} = 10x$ $\frac{\partial^2 y}{\partial z \partial x} = 10z + 10x$

(vi) $y = \frac{10z}{x^2} = 10zx^{-2}$

$\frac{\partial y}{\partial x} = -20zx^{-3} = \frac{-20z}{x^3}$ $\frac{\partial^2 y}{\partial x^2} = 60zx^{-4} = \frac{60z}{x^4}$

$\frac{\partial^2 y}{\partial x \partial z} = -20x^{-3} = \frac{-20}{x^3}$

$\frac{\partial y}{\partial z} = 10x^{-2} = \frac{10}{x^2}$ $\frac{\partial^2 y}{\partial z^2} = 0$

$\frac{\partial^2 y}{\partial z \partial x} = -20x^{-3} = \frac{-20}{x^3}$

2) For each of the following functions, find the own- first partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

(a) $z = 5x^4 + 3x^2 y + y^2$

$\frac{\partial z}{\partial x}$= 20x^3 + 6xy Note that in the $3x^2y$ term the *y* is treated as a multiplicative constant, hence re-appears in the derivative; but the $+y^2$ term is an additive constant, hence disappears.

$\frac{\partial z}{\partial y}$= 3x^2 + 2y Note that the $5x^4$ disappears because it is an additive constant; but in the $3x^2y$ term $3x^2$ is a multiplicative constant, hence reappears in the derivative. It multiplies the derivative of *y*, which is 1.

(b) $z = (x^2 + y^3)^{\frac{1}{2}}$

$$\frac{\partial z}{\partial x} = \tfrac{1}{2}(x^2 + y^3)^{-\frac{1}{2}}(2x) = x(x^2 + y^3)^{-\frac{1}{2}}$$ (*using implicit function rule*).

$$\frac{\partial z}{\partial y} = \tfrac{1}{2}(x^2 + y^3)^{-\frac{1}{2}}(3y^2) = \tfrac{3}{2}y^2(x^2 + y^3)^{-\frac{1}{2}}$$

(c) $z = (x^2 + y^2)^3$

$$\frac{\partial z}{\partial x} = 3(x^2 + y^2)^2(2x) = 6x(x^2 + y^2)^2$$

$$\frac{\partial z}{\partial y} = 3(x^2 + y^2)^2(2y) = 6y(x^2 + y^2)^2$$

(d) $z = \ln(x^2y^3)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2y^3}2xy^3 = \frac{2}{x} \quad ; \quad \frac{\partial z}{\partial y} = \frac{1}{x^2y^3}3x^2y^2 = \frac{3}{y}$$

3.Having found the first order partial derivatives for each of the functions above in Q2, now find $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial y \partial x}$, and $\frac{\partial^2 z}{\partial x \partial y}$ **for each of the functions**

Note second cross-partial derivatives are the same, so in practice you just need to find one of them! $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

(a) $z = 5x^4 + 3x^2y + y^2$

$$\frac{\partial^2 z}{\partial x^2} = 60x^2 + 6y$$

$$\frac{\partial^2 z}{\partial y \partial x} = 6x$$

$$\frac{\partial^2 z}{\partial y^2} = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x$$

(b) $z = (x^2 + y^3)^{\frac{1}{2}}$

$$\frac{\partial^2 z}{\partial x^2} = x(-\tfrac{1}{2}(x^2 + y^3)^{-\frac{3}{2}})(2x) + (x^2 + y^3)^{-\frac{1}{2}}(1)$$ *(using product and implicit function rule)*

$$= -x^2(x^2 + y^3)^{-\frac{3}{2}} + (x^2 + y^3)^{-\frac{1}{2}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = x(-\tfrac{1}{2}(x^2 + y^3)^{-\frac{3}{2}})(3y^2) + (x^2 + y^3)^{-\frac{1}{2}}(0)$$

$$= -\tfrac{3}{2}xy^2(x^2 + y^3)^{-\frac{3}{2}}$$

$$\frac{\partial^2 z}{\partial y^2} = \tfrac{3}{2}y^2\left[-\tfrac{1}{2}(x^2 + y^3)^{-\frac{3}{2}}(3y^2)\right] + (x^2 + y^3)^{-\frac{1}{2}}(3y)$$

$$= -\tfrac{9}{4}y^4(x^2 + y^3)^{-\frac{3}{2}} + (x^2 + y^3)^{-\frac{1}{2}}(3y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \tfrac{3}{2}y^2\left[-\tfrac{1}{2}(x^2 + y^3)^{-\frac{3}{2}}(2x)\right] + (x^2 + y^3)^{-\frac{1}{2}}(0)$$

$$= -\tfrac{3}{2}xy^2(x^2 + y^3)^{-\frac{3}{2}}$$

(c) $z = (x^2 + y^2)^3$

$$\frac{\partial^2 z}{\partial x^2} = 6x\left[2(x^2 + y^2)(2x)\right] + (x^2 + y^2)^2(6)$$

$$= 24x^2(x^2 + y^2) + 6(x^2 + y^2)^2$$

$$\frac{\partial^2 z}{\partial y \partial x} = 6x\left[2(x^2 + y^2)(2y)\right] + (x^2 + y^2)^2(0)$$

$$= 24xy(x^2 + y^2)$$

$$\frac{\partial^2 z}{\partial y^2} = 6y\left[2(x^2 + y^2)(2y)\right] + (x^2 + y^2)^2(6)$$

$$= 24y^2(x^2 + y^2) + 6(x^2 + y^2)^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6y\left[2(x^2 + y^2)(2x)\right] + (x^2 + y^2)^2(0)$$

$$= 24xy(x^2 + y^2)$$

(d) $z = \ln(x^2y^3)$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-2}{x^2} \;;\; \frac{\partial^2 z}{\partial y \partial x} = 0$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-3}{y^2} \;;\; \frac{\partial^2 z}{\partial x \partial y} = 0$$

4. **Find the own- first partial derivatives $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ for the function:**

$$z = (u+1)(v^2 - 1)^{0.5}$$

To find $\frac{\partial z}{\partial u}$, we apply the product rule

$\frac{\partial z}{\partial u}$ = (u+1)(0) + $(v^2 - 1)^{0.5}$ (1) = $(v^2 - 1)^{0.5}$

To find $\frac{\partial z}{\partial v}$ we differentiate $z = (u+1)(v^2 - 1)^{0.5}$ by applying the product rule *[i.e. first bracket * derivative of second bracket + second bracket * derivative of first* *bracket….. note in this case, derivative of u+1 with respect to v = 0]*, **and** the implicit function rule (when we need to get the derivative of second bracket $(v^2 - 1)^{0.5}$ with respect to v)

$$\frac{\partial z}{\partial v} = (u+1)0.5(v^2-1)^{-0.5}(2v) = (u+1)v(v^2-1)^{-0.5}$$

5. Given the demand function $Q = 10 - 3P + 2P_A + 0.2Y$, where Q is the quantity demanded, P is the price of the good, P_A is the price of an alternative good A and Y is income, find

(i) the own price elasticity of demand
(ii) the cross price elasticity of demand
(iii) the income elasticity of demand

at $P = 5$, $P_A = 3$ and $Y = 150$. Comment on the economic significance of your answers.

At $P = 5$, $P_A = 3$ and $Y = 150$:
$Q = 10 - 3P + 2P_A + 0.2Y = 10 - 3(5) + 2(3) + 0.2(150) = 10 - 15 + 6 + 30 = 31$

(i) the own price elasticity of demand: $E_P = \frac{\partial Q}{\partial P} . \frac{P}{Q}$

$$\frac{\partial Q}{\partial P} = -3$$

$$E_P = (-3).\frac{P}{Q} = (-3).\frac{5}{31} = \frac{-15}{31} = -0.4839$$

Negative own price elasticity of demand implies that the function obeys the 'Law of Demand'. Since $|E_P| < 1$ the relationship between price and demand is inelastic.

(ii) the cross price elasticity of demand: $E_{PA} = \frac{\partial Q}{\partial P_A} . \frac{P_A}{Q}$

$$\frac{\partial Q}{\partial P_A} = 2$$

$$E_{PA} = (2).\frac{P_A}{Q} = (2).\frac{3}{31} = \frac{6}{31} = 0.1935$$

Positive cross price elasticity of demand implies that the two goods are substitutes. Since $|E_{PA}| < 1$ the relationship is inelastic.

(iii) the income elasticity of demand: $E_Y = \frac{\partial Q}{\partial Y}.\frac{Y}{Q}$

$\frac{\partial Q}{\partial Y} = 0.2$

$E_Y = (0.2).\frac{150}{31} = 0.9677$

Positive income elasticity of demand implies that the good is 'normal'. Since $|E_Y| < 1$ the relationship is inelastic.

Topic 7: Partial Differentiation and Production Functions –
Marginal Product of an input (K or L), Returns to an input (K or L), Returns to Scale, Homogeneity of production function, Eulers Theorem

1. For each of the following production functions

(i) find the marginal product of Labour L and of Capital K

Marginal product of an input shows change in output from a small change in that input, holding all other inputs constant. So partially differentiate Q with respect to that input (L or K) holding all other inputs (K or L) constant

(ii) comment on the returns to K and the returns to L *(hint: you need to check both the first and second partial derivatives to examine the returns to an input)*

The returns to an input (eg K) shows how the marginal product of that input (MPK) changes at different levels of that input (K). So partially differentiate the Marginal product of an input with respect to that input

(iii) show that the function is homogenous and state the degree of homogeneity

Remember: A function is Homogeneous of Degree r if, by scaling up ALL inputs by a factor λ you can express the proportionate change in output by λ^r

$f(\lambda X, \lambda Z) = \lambda^r f(X, Z) = \lambda^r Y$ where r is the degree of homogeneity

All cobb-douglas production functions are homogenous.

Quick check of degree of homogeneity in the case of Cobb-Douglas production functions:

add the powers on K and L so if $Y = K^{\alpha}L^{\beta}$ then degree of homogeneity $= \alpha + \beta$

(iv) identify whether the function exhibits increasing, decreasing or constant returns to scale

Returns to scale: shows the change in Y due to a *proportionate* change in ALL factors of production .

So if Y= f(K, L)

<u>Constant Returns to Scale</u> if $f(\lambda K, \lambda L) = \lambda f(K, L) = \lambda Y$

<u>Increasing Returns to Scale</u> if $f(\lambda K, \lambda L) > \lambda f(K, L) > \lambda Y$

<u>Decreasing Returns to Scale</u> if $f(\lambda K, \lambda L) < \lambda f(K, L) < \lambda Y$

Quick check of returns to scale in the case of Cobb-Douglas production functions: add the powers on K and L so if $Y = K^{\alpha}L^{\beta}$ then
if $\alpha + \beta = 1$ constant returns to scale,
if $\alpha + \beta > 1$ increasing returns to scale
if $\alpha + \beta < 1$ decreasing returns to scale

i.e. check if degree of homogeneity is >1 or < 1 or =1

a) $Q = K^{1/3}L^{1/3}$

i) Marginal product labour, MPL:
$f(K, L) = K^{1/3}L^{1/3}$

$\frac{\partial Q}{\partial L} = \frac{1}{3}K^{1/3}L^{-2/3}$ > 0 so an increase in L holding K constant will increase output

Marginal product Capital, MPK:

$\frac{\partial Q}{\partial K} = \frac{1}{3}K^{-2/3}L^{1/3}$ >0 so an increase in K holding L constant will increase output

ii)

MPK = $\frac{\partial Q}{\partial K} = \frac{1}{3}K^{-2/3}L^{1/3}$ So $\frac{\partial^2 Q}{\partial K^2} = -\frac{2}{9}K^{-5/3}L^{1/3} < 0$

Since this is <0, the function exhibits Diminishing returns to Capital. i.e. an increase in K, holding L constant, will increase output but at a diminishing rate with K…… or in other words, the change in output due to a change in K is smaller at higher levels of K

MPL = $\frac{\partial Q}{\partial L} = \frac{1}{3} K^{1/3} L^{-2/3}$ So $\frac{\partial^2 Q}{\partial L^2} = -\frac{2}{9} K^{1/3} L^{-5/3} < 0$

So the function exhibits Diminishing returns to Labour. i.e. an increase in L, holding K constant, will increase output but at a diminishing rate with L…… or in other words, the change in output due to a change in L is smaller at higher levels of L

iii) show that the function is homogenous and state the degree of homogeneity

$$f(\lambda K, \lambda L) = (\lambda K)^{1/3} (\lambda L)^{1/3} = \lambda^{2/3} K^{1/3} L^{1/3} = \lambda^{2/3} f(K, L)$$

so r = 2/3 and hence function has Homogeneity of degree 2/3, and the function exhibits Diminishing returns to scale

or since function is Cobb-Douglas, homogeneous of degree 1/3 + 1/3 = 2/3

iv) Returns to scale: shows the change in Y due to a *proportionate* change in ALL factors of production .

$$f(\lambda K, \lambda L) = (\lambda K)^{1/3} (\lambda L)^{1/3} = \lambda^{2/3} K^{1/3} L^{1/3} = \lambda^{2/3} Y$$

so output increases by proportionately less than λ (i.e. by $\lambda^{2/3}$) and hence the function exhibits decreasing returns to scale

Since it is a Cobb-Douglas production functions: add the powers on K and L so
1/3 + 1/3 = 2/3 < 1 hence decreasing returns to scale
or in other words, since production function is homogenous of degree 2/3 which is < 1, it exhibits decreasing returns to scale

b) $Q = K^{2/3} L$

i) Marginal product of Capital = MPK = $\frac{\partial Q}{\partial K} = \frac{2}{3} K^{-1/3} L > 0$ an increase in K holding L constant will increase Q

Marginal Product of Labour = MPL = $\frac{\partial Q}{\partial L} = K^{2/3} > 0$ an increase in L holding K constant will increase Q

ii) $\frac{\partial Q}{\partial K} = \frac{2}{3}K^{-1/3}L$ So $\frac{\partial^2 Q}{\partial K^2} = -\frac{2}{9}K^{-4/3}L < 0$

So Diminishing returns to Capital i.e. an increase in K holding L constant will increase Q, but the effect will be smaller at higher levels of K

$\frac{\partial Q}{\partial L} = K^{2/3}$ So $\frac{\partial^2 Q}{\partial L^2} = 0$

So Constant returns to Labour i.e. an increase in L holding K constant will increase Q, and the effect will be the SAME at ALL levels of L

iii) $Y = K^{2/3}L^1$

$$f(\lambda K, \lambda L) = (\lambda K)^{2/3}(\lambda L)^1 = \lambda^{5/3}K^{2/3}L^1 = \lambda^{5/3}Y$$

$r = 5/3 > 1$ so the function is homogeneous of degree 5/3

iv) from (iii) above, increasing all inputs by a factor λ will increase output by proportionately more than λ (i.e. by $\lambda^{5/3}$) and hence the function exhibits increasing returns to scale

Since it is a Cobb-Douglas production functions: add the powers on K and L so
2/3 + 3/3 = 5/3 > 1 hence increasing returns to scale
or in other words, since production function is homogenous of degree 5/3 which is >1, it exhibits increasing returns to scale

c) $Q = K^{2/3}L^{1/3}$

i) Marginal product labour, MPL:

$\frac{\partial Q}{\partial L} = \frac{1}{3}K^{2/3}L^{-2/3}$ > 0 so an increase in L holding K constant will increase output

Marginal product Capital, MPK:

$\frac{\partial Q}{\partial K} = \frac{2}{3}K^{-1/3}L^{1/3}$ >0 so an increase in K holding L constant will increase output

ii) MPK = $\frac{\partial Q}{\partial K} = \frac{2}{3} K^{-1/3} L^{1/3}$ So $\frac{\partial^2 Q}{\partial K^2} = -\frac{2}{9} K^{-4/3} L^{1/3} < 0$

Since this is <0, the function exhibits Diminishing returns to Capital. i.e. an increase in K, holding L constant, will increase output but at a diminishing rate with K…… or in other words, the change in output due to a change in K is smaller at higher levels of K

MPL = $\frac{\partial Q}{\partial L} = \frac{1}{3} K^{2/3} L^{-2/3}$ So $\frac{\partial^2 Q}{\partial L^2} = -\frac{2}{9} K^{2/3} L^{-5/3} < 0$

So the function exhibits Diminishing returns to Labour. i.e. an increase in L, holding K constant, will increase output but at a diminishing rate with L…… or in other words, the change in output due to a change in L is smaller at higher levels of L

iii) show that the function is homogenous and state the degree of homogeneity

$Y = K^{2/3} L^{1/3}$

$$f(\lambda K, \lambda L) = (\lambda K)^{2/3} (\lambda L)^{1/3} = \lambda^1 K^{2/3} L^{1/3} = \lambda Y$$

so r = 1 and hence function has Homogeneity of degree 1, and the function exhibits constant returns to scale

or since function is Cobb-Douglas, homogeneous of degree 2/3 + 1/3 = 1

iv) Returns to scale: shows the change in Y due to a *proportionate* change in ALL factors of production .

From (iii) above, increasing all inputs by a factor λ will increase output by the same proportion λ and hence the function exhibits constant returns to scale

Since it is a Cobb-Douglas production functions: add the powers on K and L so
2/3 + 1/3 = 1 hence constant returns to scale
or in other words, since production function is homogenous of degree 1, it exhibits constant returns to scale

d) $Q = KL$

i) Marginal product labour, MPL:

$\frac{\partial Q}{\partial K} = L$ > 0 so an increase in L holding K constant will increase output

Marginal product Capital, MPK:

$\frac{\partial Q}{\partial L} = K$ >0 so an increase in K holding L constant will increase output

ii) MPK = $\frac{\partial Q}{\partial K} = L$ So $\frac{\partial^2 Q}{\partial K^2} = 0$

Since this is =0, the function exhibits constant returns to Capital. i.e. an increase in K, holding L constant, will increase output but at a constant rate with K…… or in other words, the change in output due to a change in K is the SAME at ALL levels of K

MPL = $\frac{\partial Q}{\partial L} = K$ So $\frac{\partial^2 Q}{\partial L^2} = 0$

So the function exhibits constant returns to Labour. i.e. an increase in L, holding K constant, will increase output at a constant rate with L…… or in other words, the change in output due to a change in L is the SAME at ALL levels of L

iii) show that the function is homogenous and state the degree of homogeneity

$Y = KL$

$$f(\lambda K, \lambda L) = (\lambda K)^1 (\lambda L)^1 = \lambda^2 KL = \lambda^2 f(K, L)$$

so r = 2 and hence function has Homogeneity of degree 2, and the function exhibits increasing returns to scale

or since function is Cobb-Douglas, homogeneous of degree 1 + 1 = 2

iv) Returns to scale: shows the change in Y due to a *proportionate* change in ALL factors of production .

From (iii) above, increasing all inputs by a factor λ will increase output by the a proportion >λ and hence the function exhibits increasing returns to scale

Since it is a Cobb-Douglas production functions: add the powers on K and L so
1 + 1 = 2 hence increasing returns to scale
or in other words, since production function is homogenous of degree 2, it exhibits increasing returns to scale

2) What values do the numbers α and β need to take if the production function $Y = AK^{\alpha}L^{\beta}$ is to have (a) diminishing returns to K (b) diminishing returns to L (c) constant returns to scale and (d) diminishing returns to scale

(a) MPK = $\dfrac{\partial Y}{\partial K} = A.\alpha K^{\alpha-1}L^{\beta}$ > 0

For Diminishing Returns to K: $\dfrac{\partial^2 Y}{\partial K^2} = A\alpha(\alpha-1)K^{\alpha-2}L^{\beta}$ < 0

Thus, must be that $0 < \alpha < 1$for diminishing returns to K

(b) MPL = $\dfrac{\partial Y}{\partial L} = A.\beta K^{\alpha}L^{\beta-1}$ > 0

For Diminishing Returns to L: $\dfrac{\partial^2 Y}{\partial L^2} = A\beta(\beta-1)K^{\alpha}L^{\beta-2}$ < 0

Thus, must be that $0 < \beta < 1$for diminishing returns to L

c) ***Remember:*** Returns to scale shows the change in Y due to a *proportionate* change in ALL factors of production

If Y = f(K,L) Then f(λK, λL) = ?Y

Quick way to check returns to scale in Cobb-Douglas production function

$Y = AK^{\alpha}L^{\beta}$ then if $\alpha + \beta = 1$: CRS
if $\alpha + \beta > 1$: IRS
if $\alpha + \beta < 1$: DRS

then in order to exhibit Constant Returns to Scale, $\alpha + \beta = 1$

d) Decreasing Returns to Scale, then $\alpha + \beta < 1$

3. Which of the following functions are homogeneous? In each case, state the degree of homogeneity and apply Eulers Theorem. ***[hint: Eulers theorem indicates K. $\partial Y/\partial K$ + L.$\partial Y/\partial L$ = rY where r is the degree of homogeneity of the function]***

(a) $Y = K^{½} L^{½}$

$Y(\lambda K, \lambda L) = \lambda K^{½} L^{½}$

Homogenous of degree 1 (CRS)

Eulers Theorem:

$K. \partial Y/\partial K + L.\partial Y/\partial L = rY = Y$ *(since here, r = 1)*

$= K(½ K^{-½}L^{½}) +L (½ K^{½}L^{-½})$

$= K^{½} L^{½} = Y$***……. Eulers Theorem Confirmed***

(b) $Y = aX^2 + bZ$

$Y(\lambda X, \lambda Z) = a\lambda^2X^2 + b\lambda Z$

Since $\neq \lambda^r(aX^2 + bZ)$ for any value of r, the function is not homogenous

Thus, Eulers Theorem can not apply

c) $Y = aK + bL$

$Y(\lambda K, \lambda L) = a\lambda K + b\lambda L$

$= \lambda(aK + bL)$ so the function is homogenous of degree 1

Eulers Theorem:

$K. \partial Y/\partial K + L.\partial Y/\partial L$

= K(a) +L (b) = Y……. Eulers Theorem Confirmed

(c) $Y = X_1^{0.2} X_2^{0.5} X_3^{0.5}$

$Y(\lambda X_1, \lambda X_2, \lambda X_3) = \lambda^{1.2} X_1^{0.2} X_2^{0.5} X_3^{0.5}$

Homogenous of degree 1.2 (IRS)

Eulers Theorem:

$X_1. \partial Y/\partial X_1 + X_2.\partial Y/\partial X_2 + X_3.\partial Y/\partial X_3 = rY = 1.2Y$ *(since r = 1.2 here)*

$= X_1\, 0.2Y/X_1 + X_2\, 0.5Y/X_2 + X_3\, 0.5Y/X_3$

$= 1.2Y$

4. **If Y = $K^{\alpha}L^{\beta}$, (i) write out the total differential dY, expressed in terms of output Y (ii) now write out the differential in terms of the proportionate change in output, dY/Y (iii) what relationship holds between the growth rates over time of Y, K, and L?** *[hint: change in output, capital and labour over time is given as dY/dt, dK/dt and dL/dt respectively – so you need to adjust differential in part (ii) to allow for this to give proportional changes over time, or in other words, the growth rates)*

(i) Total Differential:

dY= ∂Y/∂K.dK + ∂Y/∂L.dL

$$dY = \alpha K^{\alpha-1} L^{\beta} dK + \beta K^{\alpha} L^{\beta-1} dL$$

re-writing in terms of Y

$$dY = (\alpha K^{\alpha} L^{\beta}).\frac{1}{K} dK + (\beta K^{\alpha} L^{\beta})\frac{1}{L} dL$$

$$dY = \alpha \frac{Y}{K} dK + \beta \frac{Y}{L} dL$$

(ii) writing in terms out output growth, we just divide across everywhere by Y to get

$$\frac{dY}{Y} = \frac{\alpha}{K} dK + \frac{\beta}{L} dL$$

(iii) Thus, growth rate of output:

$$\frac{dY}{dt}\frac{1}{Y} = \alpha \frac{1}{K}\frac{dK}{dt} + \beta \frac{1}{L}\frac{dL}{dt}$$

$\frac{dY}{dt}\frac{1}{Y}$ = α(growth rate of capital $\frac{dK}{dt}\frac{1}{K}$) + β(growth rate of labour $\frac{dL}{dt}\frac{1}{L}$)