Topic 4: Indices and Logarithms

Lecture Notes:
section 3.1 Indices
section 3.2 Logarithms

Jacques Text Book (edition 4):
section 2.3 & 2.4 Indices & Logarithms
Indices

Any expression written as $a^n$ is defined as the variable $a$ raised to the power of the number $n$

$n$ is called a power, an index or an exponent of $a$

e.g. where $n$ is a positive whole number,

\[
\begin{align*}
a^1 &= a \\
a^2 &= a \times a \\
a^3 &= a \times a \times a \\
a^n &= a \times a \times a \times a \ldots \ldots n \text{ times}
\end{align*}
\]
Indices satisfy the following rules:

1) where n is *positive whole* number
\[ a^n = a \times a \times a \times a \ldots n \text{ times} \]
e.g. \[ 2^3 = 2 \times 2 \times 2 = 8 \]

2) **Negative** powers…..
\[ a^{-n} = \frac{1}{a^n} \]
e.g. \[ a^{-2} = \frac{1}{a^2} \]
e.g. where \( a = 2 \)
\[ 2^{-1} = \frac{1}{2} \quad \text{or} \quad 2^{-2} = \frac{1}{2 \times 2} = \frac{1}{4} \]
3) A **Zero** power
\[ a^0 = 1 \]
e.g. \( 8^0 = 1 \)

4) A **Fractional** power
\[ a^{\frac{1}{n}} = \sqrt[n]{a} \]
e.g. \( 9^{\frac{1}{2}} = \sqrt{9} = \sqrt{9} = 3 \)
\[ 8^{\frac{1}{3}} = \sqrt[3]{8} = 2 \]
All indices satisfy the following rules in mathematical applications

Rule 1

\[ a^m \cdot a^n = a^{m+n} \]

e.g. \( 2^2 \cdot 2^3 = 2^5 = 32 \)
Rule 2

\[ \frac{a^m}{a^n} = a^{m-n} \]

e.g. \[ \frac{2^3}{2^2} = 2^{3-2} = 2^1 = 2 \]

**note:** if \( m = n \),

then \[ \frac{a^m}{a^n} = a^{m-n} = a^0 = 1 \]

**note:** \[ \frac{a^m}{a^{-n}} = a^{m-(-n)} = a^{m+n} \]

**note:** \[ \frac{a^{-m}}{a^n} = a^{-m-n} = \frac{1}{a^{m+n}} \]
Rule 3

\[(a^m)^n = a^{m \cdot n}\]

e.g. \((2^3)^2 = 2^6 = 64\)

Rule 4

\[a^n \cdot b^n = (ab)^n\]

e.g. \(3^2 \times 4^2 = (3 \times 4)^2 = 12^2 = 144\)

Likewise,

\[\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n\quad \text{if } b \neq 0\]

e.g.

\[\frac{6^2}{3^2} = \left(\frac{6}{3}\right)^2 = 2^2 = 4\]
Simplify the following using the above Rules:

1) \( b = x^{1/4} \times x^{3/4} \)
2) \( b = x^2 \div x^{3/2} \)
3) \( b = (x^{3/4})^8 \)
4) \( b = \frac{x^2 y^3}{x^4 y} \)
LOGARITHMS

A Logarithm is a mirror image of an index

If $m = b^n$ then $\log_b m = n$

The log of $m$ to base $b$ is $n$

If $y = x^n$ then $n = \log_x y$

The log of $y$ to the base $x$ is $n$

e.g.

$1000 = 10^3$ then $3 = \log_{10} 1000$

$0.01 = 10^{-2}$ then $-2 = \log_{10} 0.01$
Evaluate the following:

1) \( x = \log_3 9 \)

the log of \( m \) to base \( b = n \) then \( m = b^n \)

the log of \( 9 \) to base \( 3 = x \) then

\[ \Rightarrow 9 = 3^x \]

\[ \Rightarrow 9 = 3 \times 3 = 3^2 \]

\[ \Rightarrow x = 2 \]

2) \( x = \log_4 2 \)

the log of \( m \) to base \( b = n \) then \( m = b^n \)

the log of \( 2 \) to base \( 4 = x \) then

\[ \Rightarrow 2 = 4^x \]

\[ \Rightarrow 2 = \sqrt{4} = 4^{1/2} \]

\[ \Rightarrow x = 1/2 \]
Using Rules of Indices, the following rules of logs apply

1) \( \log_b(x \times y) = \log_b x + \log_b y \)
   e.g. \( \log_{10}(2 \times 3) = \log_{10} 2 + \log_{10} 3 \)

2) \( \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y \)
   e.g. \( \log_{10}\left(\frac{3}{2}\right) = \log_{10} 3 - \log_{10} 2 \)

3) \( \log_b x^m = m \cdot \log_b x \)
   e.g. \( \log_{10} 3^2 = 2 \log_{10} 3 \)
From the above rules, it follows that

(1) \( \log_b 1 = 0 \)
    (since => \( 1 = b^x \), hence \( x \) must = 0)
    e.g. \( \log_{10} 1 = 0 \)

and therefore,

\( \log_b \left( \frac{1}{x} \right) = - \log_b x \)
    e.g. \( \log_{10} \left( \frac{1}{3} \right) = - \log_{10} 3 \)

(2) \( \log_b b = 1 \)
    (since => \( b = b^x \), hence \( x \) must = 1)
    e.g. \( \log_{10} 10 = 1 \)

(3) \( \log_b \left( \sqrt[n]{x} \right) = \frac{1}{n} \log_b x \)
A Note of Caution:

- All logs must be to the same base in applying the rules and solving for values.

- The most common base for logarithms are logs to the base 10, or logs to the base e \((e = 2.718281\ldots)\).

- Logs to the base e are called Natural Logarithms.
  \[
  \log_e x = \ln x
  \]

If \(y = \exp(x) = e^x\),
Then \(\log_e y = x\) or \(\ln y = x\).
Features of $y = e^x$

- non-linear
- always positive
- as $x \uparrow$ get $y \uparrow$ and $\uparrow$ slope of graph (gets steeper)
Logs can be used to solve algebraic equations where the unknown variable appears as a power.

**An Example:** Find the value of $x$

$$200(1.1)^x = 20000$$

**Simplify**

divide across by 200

$$\Rightarrow (1.1)^x = 100$$

1. **to find $x$, rewrite equation so that it is no longer a power**

$$\Rightarrow \text{Take logs of both sides}$$

$$\Rightarrow \log(1.1)^x = \log(100)$$

$$\Rightarrow \text{rule 3} \Rightarrow x \cdot \log(1.1) = \log(100)$$
2. Solve for $x$

\[
x = \frac{\log(100)}{\log(1.1)}
\]

no matter what base we evaluate the logs, providing the same base is applied both to the top and bottom of the equation

3. Find the value of $x$ by evaluating logs using (for example) base 10

\[
x = \frac{\log(100)}{\log(1.1)} = \frac{2}{0.0414} = 48.32
\]

4. Check the solution

\[
200(1.1)^x = 20000
\]

\[
200(1.1)^{48.32} = 20004
\]
Another Example: Find the value of $x$

$5^x = 2(3)^x$

1. rewrite equation so $x$ is not a power

$\Rightarrow$ Take logs of both sides

$\Rightarrow \log(5^x) = \log(2 \times 3^x)$

$\Rightarrow$ rule 1 $\Rightarrow \log 5^x = \log 2 + \log 3^x$

$\Rightarrow$ rule 3 $\Rightarrow x \log 5 = \log 2 + x \log 3$

2. Solve for $x$

$x [\log 5 - \log 3] = \log 2$

rule 2 $\Rightarrow x [\log \left( \frac{5}{3} \right)] = \log 2$

$$x = \frac{\log(2)}{\log(\frac{5}{3})}$$
3. Find the value of x by evaluating logs using (for example) base 10

\[ x = \frac{\log(2)}{\log\left(\frac{5}{3}\right)} = \frac{0.30103}{0.2219} = 1.36 \]

4. Check the solution

\[ 5^x = 2(3)^x \Rightarrow 5^{1.36} = 2(3)^{1.36} \Rightarrow 8.92 \]
An Economics Example 1

\[ Y = f(K, L) = A K^\alpha L^\beta \]

\[ Y^* = f(\lambda K, \lambda L) = A (\lambda K)^\alpha (\lambda L)^\beta \]

\[ Y^* = A K^\alpha L^\beta \lambda^\alpha \lambda^\beta = Y \lambda^{\alpha+\beta} \]

\( \alpha + \beta = 1 \) **Constant Returns to Scale**

\( \alpha + \beta > 1 \) **Increasing Returns to Scale**

\( \alpha + \beta < 1 \) **Decreasing Returns to Scale**

**Homogeneous of Degree r if:**

\[ f(\lambda X, \lambda Z ) = \lambda^r f(X, Z) = \lambda^r Y \]

Homogenous function if by scaling all variables by \( \lambda \), can write \( Y \) in terms of \( \lambda^r \)
An Economics Example 2

National Income = £30,000 mill in 1964. It grows at 4% p.a.

Y = income (units of £10,000 mill)

1964: \( Y = 3 \)
1965: \( Y = 3(1.04) \)
1966: \( Y = 3(1.04)^2 \)
1984: \( Y = 3(1.04)^{20} \)

Compute directly using calculator or

Express in terms of logs and solve

1984: \( \log Y = \log\{3 \times (1.04)^{20}\} \)

\[ \log Y = \log 3 + \log\{(1.04)^{20}\} \]
\[
\log Y = \log 3 + 20 \log(1.04)
\]
evaluate to the base 10

\[
\log Y = 0.47712 + 20(0.01703)
\]

\[
\log Y = 0.817788
\]

Find the anti-log of the solution:

\[
Y = 6.5733
\]

In 1984, \( Y = £65733 \) mill
Topic 3: Rules of Indices and Logs

Some Practice Questions:

1. Use the rules of indices to simplify each of the following and where possible evaluate:

   (i) \( \frac{3^5 \cdot 3^2}{3^6} \)

   (ii) \( \frac{5^4 \cdot 6^{-2}}{5^2} \)

   (iii) \( \frac{x^6 \cdot x^{-2}}{x} \)

   (iv) \( (4x^3)^2 \)

   (v) \( \frac{xy^2}{x^2} \)

   (vi) \( \frac{15x^6}{3x^4 \cdot 5x^2} \)
2. Solve the following equations:

(i) \( \log_4 64 = x \)

(ii) \( \log_3 \left( \frac{1}{27} \right) = x \)

(iii) \( x = 4 \ln 10 \)

(iv) \( 5^x = 25 \)

(v) \( 4e^x = 100 \)

(vi) \( e^{2x-1} = 100 \)