1. [5 POINTS] Find the determinant of the following matrix:

\[
A = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 \\
-4 & \frac{11}{7} & -3 & -2 \\
-2 & -\frac{17}{11} & 1 & -6 \\
-5 & -\frac{7}{3} & 1 & -8
\end{bmatrix}
\]

**Solution:** By Laplace expansion. Pick the first row. In that case \(|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13} + a_{14} \cdot C_{14}\). But \(a_{11} = a_{13} = a_{14} = 0\), and \(a_{12} = \frac{1}{2}\), so the expression reduces to

\[
|B| = \frac{1}{2} C_{12}
\]

Now, the minor of the element \(a_{12} = \frac{1}{2}\) is \(M_{12} = \begin{vmatrix}
-4 & -3 & -2 \\
-2 & 1 & -6 \\
-5 & 1 & -8
\end{vmatrix} = -40\), so the cofactor is \(C_{12} = (-1)^{1+2} \cdot M_{33} = -40\), and the determinant is \(|B| = 20\).
2. [8 POINTS] Consider the following Input-Output coefficient table:

<table>
<thead>
<tr>
<th>Input</th>
<th>Agriculture</th>
<th>Manufacturing</th>
<th>Services</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
</tr>
<tr>
<td>Manufacturing</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
</tr>
<tr>
<td>Services</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>Other Sources</td>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>Total</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Suppose the final demand for each of the three sectors (i.e., agriculture, manufacturing and services, respectively) is given by the vector \( \mathbf{b} = \begin{bmatrix} 20 \\ 95 \\ 85 \end{bmatrix} \). Find the vector of production, \( \mathbf{x} \).

**Solution:** The vector of production satisfies the expression

\[
(I - A) \mathbf{x} = \mathbf{b}
\]

or alternatively:

\[
\mathbf{x} = (I - A)^{-1} \mathbf{b}
\]

where \( (I - A)^{-1} \) is the Leontieff input-output inverse. One option is to find the inverse, but, since the question is NOT asking for it, I could solve for \( \mathbf{x} \) by using Cramer’s rule in equation (1).

First, we identify the matrix of inter-industry coefficients, which is

\[
A = \begin{bmatrix} 0.1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.3 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}
\]
\[
I - A = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} - \begin{bmatrix}
0.1 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.3 \\
0.1 & 0.2 & 0.1
\end{bmatrix} = \begin{bmatrix}
0.9 & -0.2 & -0.2 \\
-0.2 & 0.8 & -0.3 \\
-0.1 & -0.2 & 0.9
\end{bmatrix}.
\]

In turn, the determinant is \( |I - A| = 0.528 \).

Defining the vector of production as \( x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \), then, using Cramer's rule:

\[
x_1 = \frac{1}{|I - A|} \begin{vmatrix} 20 & -0.2 & -0.2 \\ 95 & 0.8 & -0.3 \\ 85 & -0.2 & 0.9 \end{vmatrix} = \frac{52.8}{0.528} = 100
\]

\[
x_2 = \frac{1}{|I - A|} \begin{vmatrix} 0.9 & 20 & -0.2 \\ -0.2 & 95 & -0.3 \\ -0.1 & 85 & 0.9 \end{vmatrix} = \frac{105.6}{0.528} = 200
\]

\[
x_3 = \frac{1}{|I - A|} \begin{vmatrix} 0.9 & -0.2 & 20 \\ -0.2 & 0.8 & 95 \\ -0.1 & -0.2 & 85 \end{vmatrix} = \frac{79.2}{0.528} = 150
\]

So the vector of production is \( x = \begin{bmatrix} 100 \\ 200 \\ 150 \end{bmatrix} \).

NOTE: This was a question for a Schol student. You could also solve the problem finding the inverse, but how long would it take you to find it, given that it is a matrix of order 3?
3. [8 POINTS] Define the following concepts:

(a) Continuous function

(b) Derivative function

(c) Hessian matrix

(d) Limit of a function

See lecture notes
4. [5 POINTS] Find the linear and the quadratic approximations about \( x = 1 \) for:

\[
h(x) = \frac{x^a - x^b}{x^a + x^b}
\]

where \( a > b > 0 \).

Solution: The linear approximation around \( x = 1 \) is:

\[
h(x) \approx h(1) + h'(1)(x - 1)
\]

where:

- \( h(1) = 0 \)
- \( h'(x) = \frac{1}{(x^a + x^b)} \left[ (ax^{a-1} - bx^{b-1})(x^a + x^b) - (x^a - x^b)(ax^{a-1} + bx^{b-1}) \right] = \frac{1}{(x^a + x^b)} (a - b)x^{a+b-1} \)
- Then \( h'(1) = \frac{a-b}{4} \)
- Hence, \( h(x) \approx \frac{a-b}{4}(x - 1) \)

The quadratic approximation around \( x = 1 \) is:

\[
h(x) \approx h(1) + h'(1)(x - 1) + \frac{1}{2}h''(1)(x - 1)^2
\]

so we only need to find the THIRD term

- \( h''(x) = -2(a - b)x^{a+b-1}(x^a + x^b)^{-3} + (a - b)(a + b - 1)x^{a+b-2} \)
- Then: \( h''(1) = -2(a - b)2^{-3} + (a - b)(2)^{-2}(a + b - 1) = \frac{1}{4}(a - b)(a + b - 2) \)
- Hence, \( h(x) \approx \frac{a-b}{4}(x - 1) + \frac{1}{8}(a - b)(a + b - 2)(x - 1)^2 \)
5. [12 POINTS] Consider the following equations:

\[ z = x^2 + 2xy + y^2 \]

\[ x = e^{s-t} \]

\[ y = e^{t-s} \]

(a) Using the chain rule, find the (first) derivative of \( z \) with respect to \( s \) and of \( z \) with respect to \( t \).

**Solution:**

\[ \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = 2(x+y)(e^{s-t}) + 2(x+y)(-e^{t-s}) = 2(x^2 - y^2) = 2(e^{2(s-t)} - e^{2(t-s)}) \]

\[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = 2(x+y)(-e^{s-t}) + 2(x+y)(e^{t-s}) = 2(y^2 - x^2) = 2(e^{2(t-s)} - e^{2(s-t)}) \]
(b) Using your answers in part (a), find the Hessian matrix of \( z \) in terms of \( s \) and \( t \). Show that it is a singular matrix.

**Solution:** We need four derivatives (but if you state Young’s theorem, just three):

- \( \frac{\partial^2 z}{\partial s^2} = 2 \left( (2) e^{2(t-s)} - (-2) e^{2(s-t)} \right) = 4 \left( e^{2(s-t)} + e^{2(t-s)} \right) \)

- \( \frac{\partial^2 z}{\partial s \partial t} = 2 \left( (-2) e^{2(s-t)} - (2) e^{2(t-s)} \right) = -4 \left( e^{2(s-t)} + e^{2(t-s)} \right) \)

- \( \frac{\partial^2 z}{\partial t^2} = 2 \left( (2) e^{2(t-s)} - (-2) e^{2(s-t)} \right) = 4 \left( e^{2(s-t)} + e^{2(t-s)} \right) \)

- \( \frac{\partial^2 z}{\partial s \partial t} = 2 \left( (-2) e^{2(t-s)} - (2) e^{2(s-t)} \right) = -4 \left( e^{2(s-t)} + e^{2(t-s)} \right) \) (so Young’s theorem is verified)

So the Hessian is:

\[
H = \begin{bmatrix}
4 \left( e^{2(s-t)} + e^{2(t-s)} \right) & -4 \left( e^{2(s-t)} + e^{2(t-s)} \right) \\
-4 \left( e^{2(s-t)} + e^{2(t-s)} \right) & 4 \left( e^{2(s-t)} + e^{2(t-s)} \right)
\end{bmatrix}
\]

and it is easy to verify that its determinant is zero, so it is a singular matrix.
6. [22 POINTS] Consider a firm that produces two products at a monthly cost \( C(x, y) = \frac{1}{2}x^2 + Axy + \frac{1}{2}y^2 \), where \( x \) and \( y \) represent the number of units produced of each commodity, and \( A \) is a constant. Suppose also that the firm sells all its output at a price per unit \( P_x \) for product \( x \) and \( P_y \) for product \( y \).

(a) Write down the optimisation problem for the firm.

Solution: The firm chooses \( x \) and \( y \) to maximize profits given by

\[
\Pi = P_x x + P_y y - C(x, y) = P_x x + P_y y - \frac{1}{2}x^2 - Axy - \frac{1}{2}y^2
\]

(b) What are the first order conditions? Find the values of \( x \) and \( y \) that satisfy the system of equations using either the row-reduction method or Cramer’s rule.

Solution: The first order conditions are:

\[
\frac{\partial \Pi}{\partial x} = P_x - x - Ay = 0
\]

\[
\frac{\partial \Pi}{\partial y} = P_y - y - Ax = 0
\]

Let’s rearrange the system of equations:

\[
x + Ay = P_x
\]

\[
Ax + y = P_y
\]
So, the matrix of coefficients is \( B = \begin{bmatrix} 1 & A \\ A & 1 \end{bmatrix} \) and the vector of endogenous variables is \( b = \begin{bmatrix} P_x \\ P_y \end{bmatrix} \). Second, the determinant of \( B \) is \(|B| = 1 - A^2|\). Then:

\[
x = \frac{1}{|A|} \begin{vmatrix} P_x & A \\ P_y & 1 \end{vmatrix} = \frac{P_x - AP_y}{1 - A^2}
\]

\[
y = \frac{1}{|B|} \begin{vmatrix} 1 & P_x \\ A & P_y \end{vmatrix} = \frac{P_y - AP_x}{1 - A^2}
\]

(c) What are the second order conditions? In particular, find the range of values for \( A, P_x \) and \( P_y \) that determines that the solution in (b) is a maximum.

**Solution:** The second order conditions are:

\[
\frac{\partial^2 \Pi}{\partial x^2} = -1 < 0 \\
\frac{\partial^2 \Pi}{\partial y^2} = -1 < 0 \\
\frac{\partial^2 \Pi}{\partial x \partial y} = \frac{\partial^2 \Pi}{\partial y \partial x} = -A
\]

So the Hessian is: \( H = \begin{bmatrix} -1 & -A \\ -A & -1 \end{bmatrix} \), and its determinant is \(|H| = 1 - A^2|\).

For the solution in (b) to be a maximum we need \( \frac{\partial^2 \Pi}{\partial q_1^2} < 0 \), \( \frac{\partial^2 \Pi}{\partial q_2^2} < 0 \) and \(|H| > 0|\), so we need \( 1 - A^2 > 0 \) or alternatively \(-1 < A < 1 \) for the solution to be a maximum. There are no restrictions for \( P_x \) or \( P_y \) (except for the fact that they need to be positive for the problem to make sense).
Suppose now that $P_x = P_y = P$ (this is for simplicity). Further, suppose that the firm is required to produce exactly a total of $\frac{P}{1+A}$ units per month of the two products combined.

(d) Write down the Lagrangian that represents the optimisation problem for the firm.

Solution: The firm chooses $x$ and $y$ to maximize profits $\Pi = Px + Py - C(x, y)$ subject to the constraint that $x + y = \frac{P}{1+A}$. Then, the Lagrangian is:

$$Z(x, y, \lambda) = \Pi + \lambda \left( \frac{P}{1+A} - x - y \right)$$

$$= Px + Py - \frac{1}{2}x^2 - Axy - \frac{1}{2}y^2 + \lambda \left( \frac{P}{1+A} - x - y \right)$$

(e) Using the Lagrangian method, determine the first order conditions. Find the values of $x$ and $y$ that satisfy the system of equations using either the row-reduction method or Cramer’s rule.

Solution: The first order conditions are:

$$\frac{\partial Z}{\partial \lambda} = \frac{P}{1+A} - x - y = 0$$
$$\frac{\partial Z}{\partial x} = P - x - Ay - \lambda = 0$$
$$\frac{\partial Z}{\partial y} = P - y - Ax - \lambda = 0$$
Let’s rearrange the system of equations:

\[ x + y = \frac{P}{1 + A} \]
\[ x + Ay + \lambda = P \]
\[ Ax + y + \lambda = P \]

So, the matrix of coefficients is
\[ B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & A & 1 \\ A & 1 & 1 \end{bmatrix} \]
and the vector of endogenous variables is
\[ b = \begin{bmatrix} P \\ P \\ P \end{bmatrix} . \]

Second, the determinant of \( B \) is \( |B| = 2(A - 1) \). Then:

\[ Q_1 = \frac{1}{|B|} \begin{vmatrix} \frac{P}{1+A} & 1 & 0 \\ P & A & 1 \\ P & 1 & 1 \end{vmatrix} = \frac{1}{2(A - 1)} \left( \frac{AP}{1+A} + P - \frac{P}{1+A} - P \right) = \frac{P}{2(1+A)} \]

\[ y = \frac{1}{|B|} \begin{vmatrix} 1 & \frac{P}{1+A} & 0 \\ 1 & P & 1 \\ A & P & 1 \end{vmatrix} = \frac{1}{2(A - 1)} \left( P + \frac{AP}{1+A} - P - \frac{P}{1+A} \right) = \frac{P}{2(1+A)} \]

(f) What are the second order conditions? In particular, find the range of values for \( A \) and \( P \) that determines that the solution in (e) is a maximum.
Solution: [5 points] We need to find the bordered Hessian, defined as follows:

\[ \mathcal{H} = \begin{bmatrix} 0 & \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial^2 Z}{\partial x^2} & \frac{\partial^2 Z}{\partial x \partial y} \\ \frac{\partial g}{\partial y} & \frac{\partial^2 Z}{\partial x \partial y} & \frac{\partial^2 Z}{\partial y^2} \end{bmatrix} \]

where:

- \( \frac{\partial^2 Z}{\partial x^2} = -1 \)
- \( \frac{\partial^2 Z}{\partial y^2} = -1 \)
- \( \frac{\partial^2 Z}{\partial Q_1 \partial Q_2} = -A \)
- \( \frac{\partial g}{\partial Q_1} = 1 \)
- \( \frac{\partial g}{\partial Q_2} = 1 \)

So the bordered Hessian is

\[ \mathcal{H} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & A \\ 1 & A & -1 \end{bmatrix} \]

and its determinant is \( |\mathcal{H}| = 2A + 2 \).

We need \( |\mathcal{H}| > 0 \), for the solution in (e) to be a maximum, so we just need \( A > -1 \). There are no restrictions for \( P \).