1. [5 POINTS] Find the determinant of the following matrix:

\[
A = \begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 \\
-4 & \frac{11}{7} & -3 & -2 \\
-2 & -\frac{17}{11} & 1 & -6 \\
-5 & -\frac{7}{3} & 1 & -8
\end{bmatrix}
\]

2. [8 POINTS] Consider the following Input-Output coefficient table:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Agriculture</td>
<td>Manufacturing</td>
<td>Services</td>
<td></td>
</tr>
<tr>
<td>Agriculture</td>
<td>0.1</td>
<td>0.2</td>
<td>0.2</td>
<td></td>
</tr>
<tr>
<td>Manufacturing</td>
<td>0.2</td>
<td>0.2</td>
<td>0.3</td>
<td></td>
</tr>
<tr>
<td>Services</td>
<td>0.1</td>
<td>0.2</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Other Sources</td>
<td>0.6</td>
<td>0.4</td>
<td>0.4</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td></td>
</tr>
</tbody>
</table>

Suppose the final demand for each of the three sectors (i.e., agriculture, manufacturing and services, respectively) is given by the vector

\[
b = \begin{bmatrix}
20 \\
95 \\
85
\end{bmatrix}
\]

Find the vector of production, \(x\).
3. [8 POINTS] Define the following concepts:

(a) Continuous function
(b) Derivative function
(c) Hessian matrix
(d) Limit of a function

4. [5 POINTS] Find the linear and the quadratic approximations about $x = 1$ for:

$$h(x) = \frac{x^a - x^b}{x^a + x^b}$$

where $a > b > 0$.

5. [12 POINTS] Consider the following equations:

$$z = x^2 + 2xy + y^2$$

$$x = e^{s-t}$$

$$y = e^{t-s}$$

(a) Using the chain rule, find the (first) derivative of $z$ with respect to $s$ and of $z$ with respect to $t$.

(b) Using your answers in part (a), find the Hessian matrix of $z$ in terms of $s$ and $t$. Show that it is a singular matrix.
6. [22 POINTS] Consider a firm that produces two products at a monthly cost \( C(x, y) = \frac{1}{2}x^2 + Axy + \frac{1}{2}y^2 \), where \( x \) and \( y \) represent the number of units produced of each commodity, and \( A \) is a constant. Suppose also that the firm sells all its output at a price per unit \( P_x \) for product \( x \) and \( P_y \) for product \( y \).

(a) Write down the optimisation problem for the firm.

(b) What are the first order conditions? Find the values of \( x \) and \( y \) that satisfy the system of equations using either the row-reduction method or Cramer's rule.

(c) What are the second order conditions? In particular, find the range of values for \( A \), \( P_x \) and \( P_y \) that determines that the solution in (b) is a maximum.

Suppose now that \( P_x = P_y = P \) (this is for simplicity). Further, suppose that the firm is required to produce exactly a total of \( \frac{P}{1+A} \) units per month of the two products combined.

(d) Write down the Lagrangian that represents the optimisation problem for the firm.

(e) Using the Lagrangian method, determine the first order conditions. Find the values of \( x \) and \( y \) that satisfy the system of equations using either the row-reduction method or Cramer's rule.

(f) What are the second order conditions? In particular, find the range of values for \( A \) and \( P \) that determines that the solution in (e) is a maximum.