Compound Interest

Invest €500 that earns 10% interest each year for 3 years, where each interest payment is reinvested at the same rate:

End of	interest earned	amount at end of period
Year 1	50	550 = 500(1.1)
Year 2	55	605 = 500(1.1)(1.1)
Year 3	60.5	$665.5 = 500(1.1)^3$

The interest earned grows, because the amount of money it is applied to grows with each payment of interest. We earn not only interest, but interest on the interest already paid. This is called compound interest.

More generally, we invest the principal, P, at an interest rate r for a number of periods, n, and receive a final sum, S, at the end of the investment horizon.

$$S = P(1 + r)^n$$

Example:

A principal of €25000 is invested at 12% interest compounded annually. After how many years will it have exceeded €250000?

$$10P = P(1+r)^n$$

Compounding can take place several times in a year, e.g. quarterly, monthly, weekly, continuously. This does not mean that the quoted interest rate is paid out that number of times a year!

Assume the €500 is invested for 3 years, at 10%, but now we compound quarterly:

Quarter	interest earned	amount at end of quarter
1	12.5	512.5
2	12.8125	525.3125
3	13.1328	538.445
4	13.4611	551.91

Generally:

$$S = P\left(1+\frac{r}{m}\right)^{nm}$$

where m is the amount of compounding per period n.

<u>Example:</u>

€10 invested at 12% interest for one year. Future value if compounded:

a) annuallyb) semi-annuallyc) quarterly

d) monthlye) weekly

As the interval of compounding shrinks, i.e. it becomes more frequent, the interest earned grows. However, the increases become smaller as we increase the frequency. As compounding increases to continuous compounding our formula converges to:

$$S = Pe^{rt}$$

<u>Example:</u>

A principal of €10000 is invested at one of the following banks:

a) at 4.75% interest, compounded annually

b) at 4.7% interest, compounded semi-annually

c) at 4.65% interest, compounded quarterly

d) at 4.6% interest, compounded continuously

=>

a) $10000(1.0475)$

b) $10000(1+0.047/2)^2$

c) $10000(1+0.0465)^4$

d) $10000e^{0.046t}$

Example:

Determine the annual percentage rate of interest of a deposit account which has a nominal rate of 8% compounded monthly.

$$\left(1+\frac{0.08}{12}\right)^{1*12}=1.0834$$

Example:

A firm decides to increase output at a constant rate from its current level of €50000 to €60000 during the next 5 years. Calculate the annual rate of growth required to achieve this growth.

$$50000(1+r)^5 = 60000$$

<u>Geometric Series</u>

Until now we have considered what happens to a lump-sum investment over a specific time-horizon. However, many investments occur over a prolonged period, such as life insurance, and debt is usually repaid periodically, such as with a mortgage. In order to deal with this feature of investments, we introduce *geometric series*:

Consider the following sequence of numbers:

$$2, 8, 32, 128, \ldots$$

This is called a *geometric progression* with a *geometric ratio* of four. Have we encountered anything like a geometric progression in our previous discussion?

$500(1.1), 500(1.1)(1.1), \ldots, 500(1.1)^{25}$ is a geometric progression with a geometric ratio of (1.1).

How can we make use of this? A *geometric series* is the sum of the elements of a geometric progression:

Let *a* be the first term in the progression and let *g* be the geometric ratio, and let *n* be the number of terms in the series. Then,

$$a\left(\frac{g^{n}-1}{g-1}\right)$$

Example:

A person saves €100 in a bank account at the beginning of each month. The bank offers 12% compounded monthly.

a) Determine the amount saved after 12 months.

Series of payments, each of which has a different value at the end of the investment horizon:

The final value of the 1st instalment: $100(1.01)^{12}$

The final value of the 2nd instalment: $100(1.01)^{11}$

…

The final value of the final instalment: $100(1.01)$

So:

$100(1.01)^1, 100(1.01)^2, \ldots, 100(1.01)^{12}$

Could simply add up each element in the series, …

a? *g*? *n*?

$$100(1.01)\left(\frac{(1.01)^{12}-1}{(1.01)-1}\right)=1280.93$$

b) After how many months does the amount saved first exceed €2000?

Series?

$100(1.01)^1, 100(1.01)^2, \ldots, 100(1.01)^n$

So?

$$100(1.01)\left(\frac{(1.01)^n-1}{(1.01)-1}\right)=2000$$

$$10100\left(1.01^n-1\right)=2000$$

$$\left(1.01^n-1\right)=0.198$$

$$1.01^n=1.198$$

$$n\ln(1.01)=\ln(1.198)$$

$$n=18.2$$

<u>Example:</u>

The monthly repayments needed to repay a €100,000 loan, which is repaid over 25 years when the interest rate is 8% compounded annually?

Two things happen:

Each month: repayments, so the loan shrinks.

Each year: interest applies, so the loan grows.

To see a series, write down what happens each year:

End of year Amount outstanding

1	$100{,}000(1.08) - 12x$
2	$[100{,}000(1.08) - 12x]1.08\text{-}12x$
3	$100{,}000(1.08)^3 - 12x(1.08)^2 - 12x(1.08) - 12x$

…

After 25 years:

$100{,}000^{25} - 12x\ [\ 1 + 1.08 + 1.08^2 + \ldots + 1.08^{24})$

$a = 1$

$g = 1.08$

$n = 25$

$$\frac{1.08^{25} - 1}{1.08 - 1} = 73.106$$

So, after 25 years:

$$684847.52 - 877.272\ x = 0$$

$$x = \frac{684847.52}{877.272} = 780.66$$

One thing we can take note of is the development of the loan:

End of year	Outstanding
1	98,682.08
2	97,154.73
3	95,559.18

The reduction in the size of the loan increases. Why?

Proof of Geometric Series Formula

Begin with:

$$S_n = a + ag + ag^2 + \ldots + ag^{n-1}$$

Which states that a certain amount is equal to the sum of a geometric progression containing n terms, with geometric ratio g and starting value a.

Then proceed by expressing gS_n:

$$gS_n = ag + ag^2 + ag^3 + \ldots + ag^{n-1} + ag^n.$$

Now, if we subtract S_n from rS_n we have:

$$gS_n - S_n = ag^n - a \qquad \text{|| all other terms cancel}$$

Thus, by multiplying out the common terms (S_n on the LHS & a on the RHS) and dividing across both sides by $g - 1$, we obtain:

$$S_n = a\left(\frac{g^n - 1}{g - 1}\right).$$

Q.e.d.

Investment Appraisal

Recall, if we want to know how much a specific amount of money will be given specific interest rates, we can use:

$$S = P(1+r)^n$$

or

$$S = Pe^{rt}$$

So, if we knew S rather than P, i.e. we are interested in the ***present value*** of S, we could write:

$$P = S(1+r)^{-n}$$

or

$$P = Se^{-rt}.$$

Example:

Find the present value of 1000 in four years' time if the discount rate is 10% compounded

(a) semi-annually

(b) continuously

a) $P = 1000(1.05)^{-8} = 676.84$

b) $P = 1000e^{0.1*4}$

Net Present Value

Assume the interest rate regime implied by scenario a) applies, and you are offered an alternative investment opportunity, where you invest 600 and receive 1000 after four years. Which project would you opt for?

To see this calculate the *net present value* (NPV) of the alternative: In order to receive 1000 in four years time one would have to invest 676.84 in a bank today. However, the alternative scenario offers the same nominal return if only 1000 is invested:

$$\text{NPV:} \qquad 676.84 - 600 = 76.84$$

Generally speaking, a project can be considered worthwhile if the NPV of it is positive, i.e. if I gain more from investing in it rather than in an ***equivalent*** project.

Alternatively, we could calculate the Internal Rate of Return (IRR). This is that rate of growth that, when applied to the initial outlay, yields the final sum. If this rate of growth exceeds that of alternative investments the project is worthwhile (the NPV is positive).

Example:

A project requiring an initial outlay of €15000 is guaranteed to produce a return of €20000 in three years time. Use

a) NPV

b) IRR

methods to decide whether this investment is worthwhile if the prevailing market rate is 5%, compounded annually. What if the rate were 12%?

a)

The present value of €20000 in three years time is given by:

$$P = 20000(1.05)^{-3} = 17276.75 .$$

Thus, the NPV of this project is:

$$17276.75 - 15000 = 2276.75$$

=> positive NPV.

b) For the IRR of the project,

$$20000 = 15000(1+r)^3$$

$$\Rightarrow \left(\frac{4}{3}\right)^{\frac{1}{3}} = 1 + r$$

$$\Rightarrow 0.1 = r$$

This is larger than the current interest rate of 5%.

Example:

Suppose it is possible to invest in only one of two 4 year projects, A or B. A requires an outlay of 1000 and yields 1200, while B requires an outlay of 30000 and yields 35000. If the market rate is 3% compounded annually, which of these projects is to be preferred?

$NPV_A = 1200(1.03)^{-4} – 1000 = 66.18$
$NPV_B = 35000(1.03)^{-4} – 30000 = 1097.05.$

Both are viable, but B has higher value:

Consider investing in A and placing the 29000 remaining on deposit:

$$1200 + 29000(1.03)^4 = 33.839.76.$$

Clearly project B would yield more.
What of IRR recommendations?
A

$$1200 = 1000(1+r)^4$$

$$1.2 = (1+r)^4$$

$$1.2^{1/4} = 1 + r$$

$$0.047 = r_A$$

B

$$35000 = 30000(1+r)^4$$

$$r_B = 0.039$$

Annuity

An annuity is a financial instrument that (usually) provides annual nominal payments for a certain period. A perpetuity does this for 'eternity'.

Find the present value of an annuity which yields an income of €10,000 for ten years, with an interest rate of 7% compounded annually. What about a perpetuity?

What we face is a stream of income, and we would like to know how much one should pay in order to obtain that income stream. Naturally, this arrangement should have the same value as the sum of the present values of ten individual contracts, each offering one payment in one of the ten years, i.e. we can break the series up into its constituent components, evaluate these, and add up their values, which we gain from the present value formula.

$$P = S(1+r)^{-t}$$

Thus, the vale of the first payment is given by:

$$P = 10000(1.07)^{-1} = 9345.79$$

That is, if we invested €9345.79 today, we would obtain €10,000 a year from now.

If we write this out for all ten years, we obtain a geometric progression, with starting value, a = $10000(1.07)^{-1}$, and a geometric ratio, $g = (1.07)^{-1}$. Thus,

$$10000(1.07)^{-1}\left(\frac{1.07^{-10}-1}{1.07^{-1}-1}\right) = 70235.82$$

What of a perpetuity?

N.B.: As we are discounting, the power the geometric ratio, g, is raised to is negative. This implies that for $g > 1 \Rightarrow 1 > g^{-n} > 0$ for all $n > 0$. Also, the larger is n, the closer to zero g^{-n} becomes. In the case of a perpetuity, $n = \infty$ and $g^{-\infty} = 0$.

$$S_{\infty} = a\left(\frac{0-1}{g-1}\right) = \left(\frac{-a}{g-1}\right) = \left(\frac{a}{1-g}\right)$$

Similarly, we could calculate the net present value of a business proposition that provided the same income stream for an investment of €60,000:

$$€70{,}235 - €60{,}000 = €10{,}235.82$$

What if payments are irregular?

Comparing two projects, A & B, with the following irregular income streams:

Year	Revenue A	Revenue B
1	6000	10000
2	3000	6000
3	10000	9000
4	8000	1000
Total	27000	26000

Which project should be chosen?

What if the discount rate is 11%?

Year	(A) Discounted Revenue (B)

1	5405.41	9000.01
2	2434.87	4869.73
3	7311.91	6580.72
4	5269.85	658.73
Total PDV	20422.04	21109.19

With an initial outlay of €20,000, what are the respective NPVs?

$NPV_A = 20422.04 – 20000 = 422.67$

$NPV_B = 21109.19 – 20000 = 1109.19$

<u>Example</u>

What is the IRR of a project that requires an initial outlay of €20,000 and produces a return of €8,000 at the end of year 1 and €15,000 at the end of year 2.

$$20000 = 8000(1+r)^{-1} + 15000(1+r)^{-2}$$

$$20000(1+r)^2 = 8000(1+r) + 15000$$

$$20000 + 40000r + r^2 = 8000 + 8000r + 1500$$

Since,

$$r^2 + 1.6r - 0.15 = 0$$

We can solve this quadratic for *r*.

For: $x^2 + px + q = 0$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Thus,

$$x_{1,2} = -0.8 \pm \sqrt{\frac{2.56}{4} + 0.15}$$

$$x_{1,2} = -0.8 \pm \sqrt{0.64 + 0.15}$$

$$\Rightarrow x = 8.9\%$$