Total Returns on Equity indices, Fat Tails and the
α-Stable Distribution

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1 Introduction

The use of the normal distribution is ubiquitous in statistical analysis in all branches of science. Ever since the days of Bernoulli (1654-1705), De Moivre (1667–1743), Bernoulli (1654-170), Laplace (1749–1827) and Gauss (1777-1855) it has been recognized that, subject to certain fairly unrestrictive conditions, any datum, that is the result of the aggregation of many individual data, has an approximate normal distribution. The return\(^1\) on many assets is the result of agents processing many of items of information and it may be argued that the accumulation of such information is the equivalent of many shocks to returns and the result is a normal distribution of returns.

During the 60’s and the early 70’s the normality assumption underlying various asset returns was questioned by, in particular, Mandelbrot (1962, 1964, 1967, 1997), Mandelbrot and Hudson (2004) and Fama (1964, 1965, 1976). Mandelbrot examined the variation of prices of cotton (1816-1940), wheat (1883-1936), railroad stock (1857-1936) and interest and exchange rates (similar periods) and found a larger number of extreme values than could be justified by the assumption of a normal distribution. Fama examined the distribution of daily returns for the 30 stock in the Dow Jones Industrial Average in a period from about the end of 1957 to September 26 1962. These papers offered support for the hypothesis that returns followed an \(\alpha\)-stable\(^2\) rather than a normal process. An \(\alpha\)-stable distribution may be thought of as a generalization of the normal distribution where the generalization allows greater concentration close to the mean, more extreme values and possible skewness. We will see that the normal distribution is an \(\alpha\)-stable with restricted values placed on its parameters. The application of \(\alpha\)-stable processes to modeling total returns on equity indices is the topic of this paper.

Cootner (1964b) is an interesting summary of thinking on the nature of asset prices at the time of the Mandelbrot and Fama papers. It is notable in that it contains the first English translation of Bachelier (1900). This, Bachelier’s Ph.D. thesis deals with the pricing of options in speculative markets. It contains original contributions to the theory of random walks, Wiener processes and diffusions and anticipates Einstein’s work in a different context. The thesis was rediscovered and translated more than half a century later. It anticipates many of the insights that were the foundation of the current theories in finance. Cootner (1964b) also reprints Mandelbrot (1964) and Fama (1964) as well as the following equations:

\[
R_t = 100 \log \left( \frac{P_t + D_t}{P_{t-1}} \right) 
\approx 100 \left( \frac{P_t + D_t}{P_{t-1}} - 1 \right)
\]

\(^1\)Throughout this paper the return on an asset is measured as 100 times the log difference of the asset price (including dividends). Thus if \(P_{t-1}\) and \(P_t\) are the prices of the asset in periods \(t - 1\) and \(t\), respectively, and \(D_t\) the dividend paid in period \(t\) the return \(R_t\) paid on the asset in period \(t\) given by

\[
R_t = 100 \log \left( \frac{P_t + D_t}{P_{t-1}} \right) 
\approx 100 \left( \frac{P_t + D_t}{P_{t-1}} - 1 \right)
\]

\(^2\)There is a certain confusion in the literature about the name to be given to this family of distributions. Mandelbrot used the term L-stable after Lévy. The probability literature uses the term stable which is unfortunate as it implies, to the non-mathematician, properties which are not appropriate. The terms \(\alpha\)-stable, stable Paretoian, stable Pareto or even Pareto-Levy are also used. Here I use the terms \(\alpha\)-stable or stable to denote this family of distributions.
as articles by prominent statisticians and economists who take the opposite view. After
this initial period of interest, work on stable processes decreased.

There were two likely reasons for the waning interest in stable distributions. First
the assumption of an underlying normal distribution had contributed, or was about to
contribute, to major breakthroughs in empirical and theoretical finance. The success
and importance of this work can be gauged by the fact that Nobel prizes have since been
awarded to Markovich, Millar, Sharpe, Merton and Scholes for their work on portfolio
allocation, Capital Asset Pricing model, Option Pricing and other contributions to the
theory of investment. The fear was, quoting Cootner (1964a)

Mandelbrot, like Prime Minister Churchill before him, promises us not utopia
but blood, sweat, toil and tears. If he is right, almost all of our statistical tools
are obsolete — least squares, spectral analysis, workable maximum-likelihood
solutions, all our established sample theory, closed distribution functions.
Almost without exception, past econometric work is meaningless....

For reasons that will become apparent the computer power necessary to work with
$\alpha$-stable distributions has only recently become available. The methods used by Man-
delbrot and Fama were ingenious and as much as could have been done at the time.
They were subject to valid criticism. It must be added that many of the criticisms that
were made were also based on similar analysis and are subject to the same criticisms.
While $\alpha$-stable distributions do provide a much better fit to the returns examined here
they still give rise to considerable implementation problems. They should be seen as
a supplement to methods utilizing normal distributions rather than as a replacement.
The problem with Cootner’ view is that he sees models arising from the normal and
$\alpha$-stable distributions as totally contradictory. The view here is that they are both
approximations to reality and that the $\alpha$-stable distribution will provide a better
approximation in many cases. Whether the value of extra effort of using $\alpha$-stable dis-
tributions is justifiable is an open question. As computer power increases the $\alpha$-stable
distribution will become easier and cheaper to use and will therefore be used more often.

A significant indication of problems with the normal distribution is that extreme events
are more frequent than the assumption of a normal distribution would predict. For
example there have been 35 falls greater than 6% in the daily Dow Jones Industrial Av-
erage since its inception in 1896, almost 110 years ago. If the changes in the (logarithm)
of the index are normally distributed on would expect that 35 falls of this magni-
tude would take place about once every 600 million years. The six total return indices con-
sidered here are available for shorter periods only but show similar discrepancies in the
numbers of large falls in the indices. For example the daily FTSE100 total returns index
which is available from 31 December 1985 shows 7 falls greater than 5% in the period to
September 2005. Assuming a normal distribution one would expect 7 such falls to occur
every 124,000 years. The daily ISEQ total returns index shows 6 falls in the period from
January 1989 to September 2005. The normality assumption would imply an expected
period of 12,000 years. The distribution of increases show similar discrepancies between the empirical distribution and the normal distribution.

There are indications here that the risk involved in such investments may be undervalued by a normality assumption. From a practical viewpoint, this is important to investment companies and to their supervisors who are measuring risk using a Value–at–Risk system based on an assumption that returns follow a normal distribution. If the element of risk is underestimated in equity price models, which assume normality, alternative models may provide some explanation of the excess equity premium paradox.

To accommodate the ‘excess’ extreme events various alternative distributions have been proposed. These have included

- replacing the normal with a t–distribution which has fatter tails than a normal distribution.
- using mixtures of normal distributions
- modeling the conditional variance of the normal distribution using GARCH methods
- using Meixner distributions (Schoutens (2005))
- using Generalized Hyperbolic Distributions (Prause (1999))
- using \( \alpha \)-Stable distributions

[Note on first 3]

[Note on next 2]

In this paper I will concentrate on the application of \( \alpha \)-stable distributions. \( \alpha \)-stable distributions have been known to mathematicians for a considerable time. According to Gut (2005) the class of \( \alpha \)-stable distributions was discovered by Paul Lévy after a lecture in 1919 by Kolmogorov, when someone told him that the normal distribution was the only possible \( \alpha \)-stable distribution. He went home and discovered that there was a family of symmetric \( \alpha \)-stable distributions the same day. Lévy’s early work is summarized in Lévy (1925) and Lévy (1954). Recent mathematical accounts are Zolotarev (1986), Samorodnitsky and Taqqu (1994) and Uchaikin and Zolotarev (1999). A summary of the main properties of the family stable distributions is given below in section 2

Section 3 analyzes six daily total returns indices (ISEQ, CAC40, DAX30, FTSE100, Dow Jones Composite and S&P500). Normal and \( \alpha \)-stable distributions are fitted to the daily returns on these indices and the fits are compared. In all cases tests of the fit of the Normal distribution are rejected. The normal distribution can be regarded as a restricted version of the \( \alpha \)-Stable distribution and the restrictions tested. In all cases the data reject these restrictions. Apart from one case the fit to the \( \alpha \)-stable distribution are acceptable. The fit to both distributions is illustrated with QQ-plots. The fit to the \( \alpha \) stable distribution is better than the fit to a normal distribution in all cases.
The presence of $\alpha$-Stable residuals in regression analysis will lead to inference problems if Ordinary Least Squares estimates are used. These problems are explained in section 4 which also gives an account of the used of maximum likelihood estimation or regression coefficients. [To illustrate week-day effects on the total return indices are estimated using both OLS and Maximum Likelihood methods. Results .......]

Perhaps the most striking facet of the $\alpha$-Stable distribution is its implication for risk measurement. This is illustrated in section 5. In general, assuming that returns follow an $\alpha$-stable process the 5% Value-at-Risk (VaR) is over estimated by on average 11% when a normal distribution is used. The 1% VaR is underestimated on average by 27%. However the 0.1% VaR is underestimated by a factor in excess of 3. [Comparisons of estimates of ETL (Expected Tail Loss) - Dowd (2002)]. [Section 6 deals with Option Pricing and $\alpha$-stable processes]
2 An Introduction to $\alpha$-Stable Processes

Some limit properties of normal random variables

This section outlines the properties of the $\alpha$-stable family of distributions and compares those with the standard normal distribution. Proofs are not given. These and further details may be found in Feller (1966), Janicki and Weron (1994), Rachev and Mittnik (2000), Samorodnitsky and Taqqu (1994), Uchaikin and Zolotarev (1999) or Zolotarev (1986).

An assumption of a normal distribution has formed part of almost all developments in theoretical and empirical finance in the last half century. In the introduction we have already referred to the Capital Asset Pricing Model, optimal portfolio allocation and option pricing as depending on a normality assumption. Some form of central\(^3\) limit theorem has been implicit in all these developments. The theorem as quoted in many econometric tests may be weakened. A version in Gut (2005) is as follows

Lindeberg-Levy-Feller Theorem: Let $X_1, X_2, \ldots$ be independent random variables with finite variances, and set, for $k \geq 1$, $E X_k = \mu_k$, $Var X_k = \sigma^2_k$, and, for $n \geq 1$, $S_n = \sum_{k=1}^{n} X_k$ and $s^2_n = \sum_{k=1}^{n} \sigma^2_k$. The Lindeberg conditions are

$$L_1(n) = \max_{1 \leq k \leq n} \frac{\sigma^2_k}{s^2_n} \to 0 \text{ as } n \to \infty$$

$$L_2(n) = \frac{1}{s^2_n} \sum_{k=1}^{n} E |X_k - \mu_k|^2 I\{|X_k - \mu_k| > \varepsilon s_n\} \to 0 \text{ as } n \to \infty$$

If equation 2 is satisfied then so is equation 1 and\(^4\)

$$\frac{1}{s^2_n} \sum_{k=1}^{n} (X_k - \mu_k) \overset{d}{\to} N(0, 1) \text{ as } n \to \infty$$

Thus the sum of independent random variables is normal subject to some fairly unrestrictive conditions on the tails of the distribution. There is even a form of inverse central limit theorem. If $Y$ has a normal distribution and $Y = X_1 + X_2$ and $X_1$ and $X_2$ are not degenerate then $X_1$ and $X_2$ have a normal distribution.

To each random variable $X$ we can assign a type \{$aX + b : a \in \mathbb{R}^+, b \in \mathbb{R}$\}. As all normal random variables can be transformed to an $N(0, 1)$ they are of one type. As the distribution of any sum of random variables, with finite variance, tends to a normal, the normal type is regarded as a domain of attraction for such random variables. We shall be ask if there are there other domains of attraction for random variables and what random variables are “attracted” to these domains of attraction?

\(^3\)The name “Central Limit Theorem” is attributed to Pólya. In the German the adjective central modifies the word theorem and not the word limit. The theorem is central to probability and statistics.

\(^4\)The notation $\overset{d}{\to}$ implies a limit in distribution. The notation $\overset{d}{=} \text{ implies that the variables on either side of the sign have the same distribution.}$
**Definition of \( \alpha \)-stable random variable**

Let \( X, X_1, X_2, \ldots \) be independent identically distributed normal random variables and let

\[
(X_1 + X_2 + \cdots + X_n) \overset{d}{=} B_n X + A_n.
\]

where

- \( B_n > 0 \) and \( A_n \) are real constants. \( A_n \) is a centralizing parameter and \( B_n \) is a normalizing factor.
- \( \lim_{n \to \infty} \max \{ P(|X_j B_n^{-1}| > \epsilon) : j = 1, 2, \ldots, n \} = 0 \). A sufficient condition for this to hold is that \( B_n \to \infty \) as \( n \to \infty \).

Then \( X \) is an \( \alpha \)-stable random variable. The term stable refers to the property that the sum of identically distributed independent random variables having the same distribution as the original up to a scale \( (B_n) \) and location factor \( (B_n) \). If the \( X \)'s are normal with mean \( \mu \) and variance \( \sigma^2 \) we may put

\[
B_n = \sqrt{n} = n^{\frac{1}{2}}
\]

\[
A_n = (1 - n^{\frac{1}{2}}) \mu
\]

to show that the normal distribution satisfies these conditions and is \( \alpha \)-stable.

It can also be shown that \( B_n \) can only take the value

\[
B_n = n^{\frac{\alpha}{2}}, \quad 0 < \alpha \leq 2.
\]

with, as shown, the value \( \alpha = 2 \) corresponding to a normal distribution. This explains the use of \( \alpha \) in the term \( \alpha \)-stable.

The characteristic function of a Stable distribution \( S \) is given by

\[
\int e^{itx}dS(x) = \begin{cases} 
\exp(-\gamma |t|\alpha [1 - i\beta(tan \frac{\pi \alpha}{2}) \text{sign } t] + i\delta t), & \text{if, } \alpha \neq 1; \\
\exp(-\gamma |t| [1 + i\beta \frac{2}{\pi} (\text{sign } t) \log(|t|)] + i\delta t), & \text{if, } \alpha = 1.
\end{cases}
\]

(see Zolotarev (1986) or Samorodnitsky and Taqqu (1994)). The \( \text{sign } t \) function is defined as

\[
\text{sign } t = \begin{cases} 
-1, & u < 0; \\
0, & u = 0; \\
1, & u > 0.
\end{cases}
\]

The distribution depends on four parameters \( \alpha, \beta, \gamma \) and \( \delta \). These parameters\(^5\) can be interpreted as follows

---

\(^5\)Note that different notation is adopted by various authorities. The principal differences include

- reversal of the sign of \( \beta \)
- Substitution of \( c = \gamma^a \)
• \( \alpha \) is the basic stability parameter. It determines the weight in the tails.

• \( \beta \) is a skewness parameter and \(-1 \leq \beta \leq 1\). A zero beta implies that the distribution is symmetric. Negative or positive \( \beta \) imply that the distribution is skewed to the left or right respectively.

• The parameter \( \gamma \) is positive and measures dispersion.

• The parameter \( \delta \) is a real number and may be thought of as a location measure.

Figures 1 to 4 illustrate various properties of \( \alpha \)-stable distributions. Figure 1 shows the density functions for symmetric (\( \beta = 0 \)) \( \alpha \) stable distributions with \( \alpha = 2 \) (normal), \( \alpha = 1.5 \) and \( \alpha = 1.0 \) (Cauchy). As \( \alpha \) is reduced note that the peak gets higher and the tails get heavier. This process continues as \( \alpha \) is reduced. Figure 2 is an enlarged version of the left tail of the distribution and shows clearly the heavier tails.

Figure 3 shows the effect of varying the symmetry parameter \( \beta \) for fixed \( \alpha \). With \( \alpha = 1.5 \) As \( \beta \) falls from 0 to \(-1\) the left tail becomes heavier relative to the right tail and the mode of the distribution shifts to the left of the mean. Similar transformations occur in the opposite direction when \( \beta \) moves from 0 to 1. The skewness caused by a particular value of \( \beta \) increases as \( \alpha \) is reduced.

Figure 4 shows the left tail of the empirical distribution of the ISEQ return data, the normal distribution with parameters from table 1, and an \( \alpha \)-stable distribution with parameters from table 2. The departures from the normal distribution are very clear as is the fit of the normal distribution.

The density function of the stable distribution may be shown to be differentiable (and continuous) on the real line. Except in three special cases the density function of the Stable distribution can not be expressed in terms of elementary functions. The special cases are:

- Substitution of \( \sqrt{2\sigma} \) for \( \gamma \)
- The characteristic function in equation 6 is not continuous at \( \alpha = 1 \). This may lead to problems in certain circumstances. If one makes the substitution

\[
d_0 = \begin{cases} \delta + \gamma \tan \frac{\pi \alpha}{2} & \alpha \neq 1 \\ \delta + \beta^2 \gamma \log \gamma & \alpha = 1 \end{cases}
\]

Following the notation of Nolan (2006) we may write the characteristic function of an \( \alpha \)-stable function as

\[
\int e^{itx} dS(x) = \begin{cases} \exp \left[-\gamma^n |t|^n \left(1 + i\beta \frac{\tan \frac{\pi \alpha}{2}}{\tan \frac{\pi}{2}} \right) \left(\frac{|u|}{|t|} \log (|t|) + i\delta_0 t\right)\right] & \alpha \neq 1 \\ \exp \left[-\gamma |t| \left(1 + i\beta \frac{\tan \frac{\pi \alpha}{2}}{\tan \frac{\pi}{2}} \right) \log (|t|)\right] + i\delta_0 t & \alpha = 1 \end{cases}
\]

Because of the better behavior of this parametrization at \( \alpha = 1 \) it is the form most often used in numerical calculations. Nolan refers to this as an \( S(\alpha, \beta, \gamma, \delta; 0) \) distribution. The parametrization in equation 6 is referred to as an \( S(\alpha, \beta, \gamma, \delta; 1) \) distribution and is the form most often used here. In the \( S(\alpha, \beta, \gamma, \delta; 1) \) note that when \( 0 < \alpha < 1 \) \( E X = \mu \). In the \( S(\alpha, \beta, \gamma, \delta; 0) \) this does not hold, in general. Note than if \( \beta = 0 \) or \( \alpha = 2 \) the two parameterizations coincide. Here we shall use the \( S(\alpha, \beta, \gamma, \delta; 0) \) parametrization and the density and distribution functions will be denoted by \( s(x, \alpha, \beta, \gamma, \delta) \) and \( S(x, \alpha, \beta, \gamma, \delta) \) respectively. If the variables are standardized (\( \gamma = 1 \) and \( \delta = 0 \) we may use the symbols \( s(x, \alpha, \beta) \) and \( S(x, \alpha, \beta) \) for the density and distribution.
Normal Description If $\alpha = 2$ the characteristic function in equation (6) reduces to
\[ \phi(it) = \int e^{itx}dH(x) = \exp(i\delta t + \gamma^2 t^2) \] (8)
Which is the characteristic function of a normal distribution
\[ \frac{1}{\gamma \sqrt{\pi}} \exp \frac{(x-\delta)^2}{\gamma^2}, \quad -\infty < x < \infty \]
with mean $\delta$ and variance $2\gamma^2$. Note that the symmetry parameter does not appear in the characteristic function in this case.

Cauchy Distribution When $\alpha = 1$ and $\beta = 0$ the characteristic function reduces to
\[ \exp(-\gamma |t| + i\delta t) \]
which is the characteristic function of the Cauchy Distribution
\[ \frac{1}{\pi(\gamma^2 + (x-\delta)^2)}, \quad -\infty < x < \infty \]

Levy Distribution When $\alpha = 1/2$ and $\beta = -1$ the distribution becomes a Levy distribution
\[ \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp \left(-\frac{\sigma}{2(x-\mu)}\right), \quad \mu < x < \infty \]

Generalized Central Limit Theorem – Domains of attraction
Consider a random variable $X$ with density function such that
\[ F(x) \sim \begin{cases} B_- |x|^{-(1+a)} & \text{as } x \to -\infty \\ B_+ |x|^{-(1+a)} & \text{as } x \to \infty \end{cases} \] (9)
where $0 < a < 2$. Thus the tails of the distribution have an asymptotic Pareto distribution. Put
\[ b = \frac{B_+ - B_-}{B_+ + B_-} \] (10)

The Pareto distribution was used by Pareto almost on hundred years ago to model the distribution of incomes above a certain threshold. A random variable has a Pareto distribution if its density function is of the form:
\[ f_X(x;a,b) = ab^n x^{-(1+a)} \quad x > b, \quad a > 0, \quad b > 0 \]
This distribution has a remarkable property known as scaling. If we increase the threshold the shape of the distribution remains the same apart from a scaling factor. For example, by integration, $P[X \geq cb] = c^{-a}$. Then the distribution of $X$ given that $X \geq cb$, where $c > 1$ is given by
\[ f_X(x;a,b) = a(cb)^n x^{-(1+a)} \quad x > cb, \quad a > 0, \quad b > 0, \quad c > 1 \]
Thus $P[X \geq c^2 b | X \geq 2b] = c^{-a}$. To illustrate let the distribution of the wealth of persons with wealth greater than say €1,000,000 be Pareto with parameter $a = 1.5$. Then the probability that a person in this group will have wealth of twice the threshold is about 0.35. Now let the threshold be €2,000,000 then the probability that a person above that threshold will have a wealth of twice that threshold (€4,000,000) is again 0.35. This is in complete contrast to the normal or lognormal distribution. Note that the mean of this distribution exists if $a > 1$ and the variance if $a > 2$. 


Then if $X_1, X_3, \ldots, X_n$ are independent, identically distributed random variables with this asymptotic distribution then the random variable

$$S = \frac{1}{n^a} \sum_{i=1}^{n} X_i$$  \hspace{1cm} (11)

has a limit in distribution which is $\alpha$-stable with parameters $\alpha = a$ and $\beta = b$.

Thus each member of the family of $\alpha$-stable distributions possesses a domain of attraction. This domain includes all distributions with the Pareto tails described in equation 9.

**Some properties of $\alpha$-stable distributions**

Some of the more important properties of $\alpha$-stable distributions are given below:

- The only $\alpha$-stable distribution for which moments of all orders exist is the normal distribution. When $1 < \alpha < 2$ the variance is not defined (infinite) and only the mean exists. In our notation the mean is given by $EX = \delta$. Apart from the lack of a simple form for the density function of an $\alpha$-stable density function the non-existence of a variance is the greatest barrier to their use. Put simply measures of the variance of an $\alpha$-stable process will increase with sample size and will not converge.

If $0 < \alpha \leq 1$ the mean does not exist. If $\alpha < 1$ the mean is even more dispersed than the individual measurements. In applications of $\alpha$-stable distributions to finance values of $\alpha$ are usually of the order of 1.5 to 1.8 are usually appropriate. The values estimated in section 3 vary from 1.65 to 1.73.

- The $\alpha$-stable density is symmetric with respect to simultaneous changes of the sign of $x$ and $\beta$, that is

$$s(x, \alpha, \beta, \gamma, \delta) = s(-x, -\beta, \gamma, \delta)$$  \hspace{1cm} (12)

- If $a$ and $b > 0$ are real constants then the density of $\frac{x-a}{b}$ is given by

$$\frac{1}{b} s\left(\frac{x-a}{b}, \alpha, \beta, \frac{\gamma}{b}, \frac{\delta-a}{b}\right)$$  \hspace{1cm} (13)

or in particular that of $\frac{x-\delta}{\gamma}$ by

$$\frac{1}{\gamma} s\left(\frac{x-\delta}{\gamma}, \alpha, \beta, 1, 0\right)$$  \hspace{1cm} (14)

where $s(x, \alpha, \beta, \gamma, \delta)$ is the density of $x$. $\delta$ and $\gamma$ are described as location and scale parameters respectively.

- Let $X_1$ and $X_2$ be $\alpha$-stable random variables with densities $s(x, \alpha, \beta_i, \gamma_i, \delta_i), \ i = 1, 2$. Then $X_1 + X_2$ is $\alpha$-stable with

$$\beta = \frac{\beta_1 \gamma_1^\alpha + \beta_2 \gamma_2^\alpha}{\gamma_1^\alpha + \gamma_2^\alpha}, \quad \gamma = (\gamma_1^\alpha + \gamma_2^\alpha)^{\frac{1}{\alpha}}, \quad \delta = \delta_1 + \delta_2$$  \hspace{1cm} (15)
In general use, the density functions of an $\alpha$-stable process may be estimated by an inverse numerical transform of the characteristic function. For some purposes the numerical integration routines in Mathematica (Wolfram (2003)) may be sufficient. To provide greater accuracy in the tails of the distribution some form of series or integral expansion of the characteristic function is often used. Programs to compute $\alpha$-stable density and distribution functions are available in Mathematica (Rimmer (2005)), R (Wuertz (2005)) or as the stand-alone program STABLE (Nolan (2005)). The calculations in this paper make considerable use of these packages.
3 Comparison of fit of Normal and $\alpha$-Stable distributions to Return Distributions

Table 1 gives summary statistics for each of the six total returns indices. The most notable feature of these statistics are the extreme estimates of the kurtosis of returns. This is an indication of the long tails in the data. The table also estimates three statistics which function as goodness of fit tests to the Normal Distribution.

Jarque and Bera (JB) test The JB test is a joint test for skewness and excess kurtosis relative to a normal distribution. Given a data sample $\{x_i, \ i = 1, \ldots, N\}$ with mean $\bar{x}$ the JB test statistic is estimated as follows:

\[
\hat{\sigma}^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2
\]

\[
m_k = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^k
\]

\[
Sk = \frac{N^2}{(N-1)(N-2)} \frac{m_3}{\hat{\sigma}^3} \quad \text{(Skewness)}
\]

\[
Ku = \frac{N^2}{(N-1)(N-2)(N-3)} \frac{(N+1)m_4 - 3(N-1)m_2^2}{\hat{\sigma}^4} \quad \text{(Kurtosis)}
\]

\[
JB = N \left( \frac{(Ku)^2}{24} + \frac{(Sk)^2}{6} \right) \quad \text{(Jarqu-Bera statistic)}
\]

Under the assumption that $x_i$ is a random sample from a normal distribution the Jarque-Bera statistic is asymptotically $\chi^2$ with 2 degrees of freedom. In this case the statistics indicate very significant departures from a normal distribution.

Kolmorogov Smirnov (KS) test The KS test in this case compares the cumulative sample distribution $S(x)$ defined as the proportion of the sample that is less than $x$ with that of the the hypothesized distribution $F(x)$ and is given by

\[
KS = \text{Max}|F(x) - S(x)|
\]

Critical values for the KS statistic for larger samples are given in Maraglia et al. (2003). The values here indicate very significant departures from normality.

Shapiro Wilk (SW) test The SW test is base on the correlation between $x(i)$ and $F^{-1}(i/N+1)$ which are the $i/n$ quantiles of the sample and population respectively. Again the normal hypothesis is rejected and the conclusions following the JB and KS statistics are confirmed.

Figures 5, 6 and 7 are normal QQ-plots of the return data for each index. A QQ-plot is a diagnostic plot designed to show the closeness of two distributions. Both distributions can be empirical when the aim is to look at the similarity of the two empirical distributions, In this case one distribution is the relevant return distribution and the other is a normal distribution with the same mean and variance. This normal QQ-plot plots the empirical quantiles of the data against the corresponding quantiles of
Table 1: Summary Statistics Equity Total Returns and their fit to a Normal Distribution

<table>
<thead>
<tr>
<th></th>
<th>ISEQ</th>
<th>CAC40</th>
<th>DAX 30</th>
<th>FTSE100</th>
<th>DJC</th>
<th>S&amp;P500</th>
</tr>
</thead>
<tbody>
<tr>
<td>start date</td>
<td>04/01/88</td>
<td>31/12/87</td>
<td>28/09/59</td>
<td>31/12/85</td>
<td>30/09/87</td>
<td>03/01/89</td>
</tr>
<tr>
<td>end date</td>
<td>21/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
</tr>
<tr>
<td>observations</td>
<td>4622</td>
<td>4627</td>
<td>12000</td>
<td>5149</td>
<td>4693</td>
<td>4363</td>
</tr>
<tr>
<td>mean</td>
<td>0.052</td>
<td>0.044</td>
<td>0.022</td>
<td>0.041</td>
<td>0.038</td>
<td>0.043</td>
</tr>
<tr>
<td>St. dev</td>
<td>0.934</td>
<td>1.277</td>
<td>1.148</td>
<td>1.028</td>
<td>1.007</td>
<td>0.980</td>
</tr>
<tr>
<td>Skewness</td>
<td>-0.3634</td>
<td>-0.124</td>
<td>-0.282</td>
<td>-0.732</td>
<td>-2.686</td>
<td>-0.198</td>
</tr>
<tr>
<td>JB test(^a)</td>
<td>5690</td>
<td>1749</td>
<td>35254</td>
<td>21123</td>
<td>667907</td>
<td>3362</td>
</tr>
<tr>
<td>KS test(^b)</td>
<td>0.065</td>
<td>0.054</td>
<td>0.062</td>
<td>0.055</td>
<td>0.074</td>
<td>0.063</td>
</tr>
<tr>
<td>SW test(^c)</td>
<td>0.941</td>
<td>0.967</td>
<td>NA</td>
<td>NA</td>
<td>0.8689</td>
<td>0.956</td>
</tr>
</tbody>
</table>

\(^a\) The asymptotic distribution of the Jarque-Bera statistic is \(\chi^2(2)\) with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.

\(^b\) For the sample sizes here the 1% critical value for the Kolmogorov-Smirnov statistic is less that .02. See Marsaglia et al. (2003)

\(^c\) The 5% critical level for the Shapiro Wilk test is .9992 for a sample of 4500. The smaller values reported here indicate very significant departures from normality.

A normal distribution. These plots also contain distribution free 95% confidence intervals for the empirical quantiles (see Hogg et al. (2005)) and the straight line along which the QQ-plot would lie if the match were perfect. Of particular interest are the regions where the line lies outside the 95% bands. (Note that one might expect of the order of 5% of the points on the QQ curve (231 for ISEQ) to lie outside these bands). All 6 plots show considerable deviations from the normal distribution. They illustrate the considerable weight in the tails of the distribution relative to the normal as would be expected from the kurtosis statistics in table 1. The graphs also show problems near the center of the distributions. All show excess concentration of returns in the empirical distribution relative to the normal at the center of the distribution.

Table 2 gives maximum likelihood estimates of the parameters of all six returns indices along with 95% confidence half width estimates. These estimates have been derived using John Nolan’s program [http://academic2.american.edu/~jpnolan/stable/stable.exe](http://academic2.american.edu/~jpnolan/stable/stable.exe). The estimates of the \(\alpha\)-stability parameter found have an average value of 1.688 a minimum of 1.646 for the ISEQ and a maximum of 1.726 for the FTSE. On the basis of the estimated half width confidence intervals all values are significantly different from 2. The CAC, DAX and FTSE indices show significant negative skew. The ISEQ, Dow Jones Composite and the S&P also show negative skew but it is not significant. By applying the restrictions \(\alpha = 2\) and \(\beta = 0\) and re-estimating one can complete a likelihood ratio test of the restrictions. The restrictions are rejected at very small confidence levels. A KS test for goodness of fit to the stable distribution is accepted for all except the S&P500.
Table 2: Estimates of Parameters of Stable distributions of Equity Total Return Indices (complete period)

<table>
<thead>
<tr>
<th></th>
<th>ISEQ</th>
<th>CAC40</th>
<th>DAX 30</th>
<th>FTSE100</th>
<th>DJC</th>
<th>S&amp;P500</th>
</tr>
</thead>
<tbody>
<tr>
<td>start date</td>
<td>04/01/88</td>
<td>31/12/87</td>
<td>28/09/59</td>
<td>31/12/85</td>
<td>30/09/87</td>
<td>03/01/89</td>
</tr>
<tr>
<td>end date</td>
<td>21/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
<td>26/09/05</td>
</tr>
<tr>
<td>observations</td>
<td>4622</td>
<td>4627</td>
<td>12000</td>
<td>5149</td>
<td>4693</td>
<td>4364</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.646 (0.045)</td>
<td>1.718 (0.043)</td>
<td>1.687 (0.027)</td>
<td>1.726 (0.041)</td>
<td>1.684 (0.044)</td>
<td>1.668 (0.046)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-0.064 (0.111)</td>
<td>-0.147 (0.128)</td>
<td>-0.076 (0.075)</td>
<td>-0.147 (0.125)</td>
<td>-0.076 (0.119)</td>
<td>-0.105 (0.118)</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.502 (0.014)</td>
<td>0.746 (0.020)</td>
<td>0.627 (0.011)</td>
<td>0.583 (0.015)</td>
<td>0.529 (0.015)</td>
<td>0.550 (0.017)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.054</td>
<td>0.032</td>
<td>0.019</td>
<td>0.036</td>
<td>0.042</td>
<td>0.034</td>
</tr>
<tr>
<td>KS (stable)</td>
<td>0.012</td>
<td>0.014</td>
<td>0.010</td>
<td>0.008</td>
<td>0.018</td>
<td>0.023</td>
</tr>
<tr>
<td>p-value</td>
<td>0.518</td>
<td>0.307</td>
<td>0.166</td>
<td>0.892</td>
<td>0.097</td>
<td>0.025</td>
</tr>
<tr>
<td>LR$^b$ test of Normality</td>
<td>838.1</td>
<td>418.6</td>
<td>1945.8</td>
<td>786.7</td>
<td>1236.5</td>
<td>583.0</td>
</tr>
</tbody>
</table>

$^a$ Figures in brackets under each coefficient estimate are the 95% confidence interval half width estimates

$^b$ Likelihood ratio test of the joint restriction $\alpha = 2$ and $\beta = 0$. The test statistic is asymptotically $\chi^2(2)$ with critical values 5.99 and 9.21 at the 5% and 1% levels respectively.
Figures 8, 9 and 10 are stable QQ-plots of the return data for each of the six returns indices. The construction of these curves is similar to that of the normal QQ-plots except that the normal distribution is replaced by the $\alpha$-stable distribution with parameters taken from table 2. The fit for the European indices is excellent, almost too good. At the extremes there is a suggestion that the tails of the empirical distribution are a little lighter than the theoretical stable distribution but this is not significant. What is surprising is that the fit in the center of the distribution is so superior to that of the normal distribution. The fit of the American indices to the stable distribution is again far superior to the normal distribution. There are some deviations in the center of the American distribution which are not obvious in the diagrams. These are very small relative to the deviations from the normal distribution but might be the subject of further work.
4 Regression with non-normal $\alpha$-Stable Errors

Consider the standard regression model

$$y_i = \sum_{j=1}^{k} x_{ij} \beta_j + \varepsilon_i, \quad i = 1, \ldots, N$$

(16)

where $y_i$ is an observed dependent variable, the $x_{ij}$ are observed independent variables, $\beta_j$ are unknown coefficients to be estimated and $\varepsilon_i$ are identically and independently distributed. Equation 16 may be written in matrix form as

$$y = X \beta + \varepsilon$$

(17)

where

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \quad X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1k} \\ x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{Nk} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix}$$

(18)

The standard OLS estimator of $\beta$ is

$$\hat{\beta} = X'X^{-1}X'y$$

(19)

Thus

$$\hat{\beta}_{OLS} - \beta = X'X^{-1}X'\varepsilon$$

(20)

Thus in the simplest case where $X$ is predetermined $\hat{\beta}_{OLS} - \beta$ is a linear sum of the elements of $\varepsilon$. If the elements of $\varepsilon$ are independent identically distributed non-normal $\alpha$-stable variables then $\hat{\beta}_{OLS}$ has an $\alpha$-stable distribution. The variance of $\varepsilon_i$ does not even exist. Thus standard OLS inferences are not valid. (Logan et al. (1973)) prove the following properties of the asymptotic t-statistic

1. The tails of the distribution function are normal-like at $\pm \infty$

2. The density has infinite singularities $|1 \mp x|^{-\alpha}$ at $\pm 1$ for $0 < \alpha < 1$ and $\beta \neq \pm 1$.

   When $1 < \alpha < 2$ the distribution has peaks at $\pm 1$.

3. As $\alpha \to 2$ the density tends to normal and the peaks vanish

When $1 < \alpha < 2$ the OLS estimates are consistent but converge at a rate of $n^{\frac{\alpha}{2} - 1}$ rather than $n^{-\frac{1}{2}}$ in the normal case.

DuMouchel (1971, 1973, 1975) shows that, subject to certain conditions, the maximum likelihood estimates of the parameters of an $\alpha$-stable distribution have the usual asymptotic properties of a Maximum Likelihood estimator. They are asymptotically normal, asymptotically unbiased and have an asymptotic covariance matrix $n^{-1} I(\alpha, \beta, \gamma, \delta)^{-1}$.
where $I(\alpha, \beta, \gamma, \delta)$ is Fisher’s Information. McCulloch (1998) examines linear regression in the context of $\alpha$-stable distributions paying particular attention to the symmetric case. Here the symmetry constraint is not imposed.

Assume that $\varepsilon_i = y_i - \sum_{j=1}^{k} x_{ij} \beta_j$ is $\alpha$-stable with parameters $\{\alpha, \beta, \gamma, 0\}$. If we denote the $\alpha$-stable density function by $s(x, \alpha, \beta, \gamma, \delta)$ then we may write the density function of $\varepsilon_i$ as

$$s(\varepsilon_i, \alpha, \beta, \gamma, \delta) = \frac{1}{\gamma} s \left( \frac{y_i - \sum_{j=1}^{k} x_{ij} \beta_j}{\gamma}, \beta, 1, 0 \right),$$

the Likelihood as

$$L(\varepsilon, \alpha, \beta, \gamma, \beta_1, \beta_2, \ldots) = \left(\frac{1}{\gamma}\right)^n \prod_{i=1}^{n} s \left( \frac{y_i - \sum_{j=1}^{k} x_{ij} \beta_j}{\gamma}, \beta, 1, 0 \right),$$

and the Loglikelihood as

$$l(\varepsilon, \alpha, \beta, \gamma, \beta_1, \beta_2, \ldots) = \sum_{i=1}^{n} \left( -n \log(\gamma) + \log \left( s \left( \frac{y_i - \sum_{j=1}^{k} x_{ij} \beta_j}{\gamma}, \beta, 1, 0 \right) \right) \right)$$

$$= \sum_{i=1}^{n} \phi(\hat{\varepsilon}_i).$$

The maximum likelihood estimators are the solutions of the equations

$$\frac{\partial l}{\partial \beta_m} = \sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) x_{im} = 0, \quad m = 1, 2, \ldots, k$$

$$\sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) \hat{\varepsilon}_i x_{im} = 0, \quad m = 1, 2, \ldots, k$$

$$\sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) (y_i - \sum_{j=1}^{k} x_{ij} \beta_j) x_{im} = 0, \quad m = 1, 2, \ldots, k$$

$$\sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) (y_i - \sum_{j=1}^{k} x_{ij} \beta_j) x_{im} = 0, \quad m = 1, 2, \ldots, k$$

$$\sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) y_{i} x_{im} = \sum_{i=1}^{n} -\phi'(\hat{\varepsilon}_i) \sum_{j=1}^{k} x_{ij} \beta_j$$

If $W$ is the diagonal matrix

$$W = \begin{pmatrix}
\frac{-\phi'(\hat{\varepsilon}_1)}{\hat{\varepsilon}_1} & 0 & \cdots & 0 \\
0 & \frac{-\phi'(\hat{\varepsilon}_2)}{\hat{\varepsilon}_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{-\phi'(\hat{\varepsilon}_n)}{\hat{\varepsilon}_n}
\end{pmatrix},$$

Using the notation in equation (18) we may write equation (24) in matrix format.

$$X' W y = (X' W X) \hat{\beta}$$

(26)
or if $X'WX$ is not singular

$$
\hat{\beta} = (X'WX)^{-1}X'Wy
$$

Thus the maximum likelihood regression estimator has the format of a Generalized Least Squares estimator in the presence of heteroscedasticity where the variance\(^7\) of the error term $\varepsilon_i$ is proportional to $\frac{\phi(\varepsilon_i)}{\varepsilon_i}$. The effect of the “Generalized Least Squares” adjustment is to give less weight to larger observations. Figure 11 compares the weighting pattern derived from equation (25) for $\alpha$-stable processes with $\alpha = 1.2$ and 1.6 with those of a standard normal distribution. For compatibility purposes the $\alpha$-stable curves are drawn with $\gamma = 1/\sqrt{2}$. As expected the normal distribution gives equal weights to all observations. The estimator for $\alpha$-stable processes gives higher weights to the center of the distribution and extremely small weights to extreme values. This effect increases as $\alpha$ is reduced.

This result explains the results obtained by Fama and Roll (1968) who completed a Monte Carlo study of the use of truncated means as measures of location in $\alpha$-stable distributions. They found

> When $\alpha = 1.1$ the .25 truncated\(^6\) means are still dominant for all $n$. For $\alpha = 1.3$ and $\alpha = 1.5$ the .50 truncated means are generally best, and when $\alpha = 1.9$ the distributions of the .75 truncated means are uniformly less disperse than those of other estimators. Finally, when the generating process is Gaussian ($\alpha = 2$) the mean is the “best” estimator. Of course it is also minimum-variance, unbiased in this case.

The shape of the weight curves in the skewed case is shown in figure (12). The weights are based on the same $\alpha$-stable distributions as those in figure 11 except that $\beta$ is now $-0.1$. The most surprising aspect of the weighting systems is the negative weights given to small positive observations. Again the effects are more pronaunces as $\alpha$ is reduced.

Empirical analysis suggests that there is a recurrent low or negative return on equities from Monday to Friday. This effect is known as the weekend effect. The existence of this effect would allow one to design a strategy to make excess profits and would have implications for the Efficient Markets Hypothesis. It is likely that, if residuals are $\alpha$-stable, then assuming normal residuals and estimating day of week effects using Ordinary Least Squares residuals will lead to spurious results. The use of $\alpha$-stable residuals and maximum likelihood will lead to a more robust result. Both methodologies\(^9\) will be applied to the entire and to subsets of each of the six total returns indices.

Estimation is carried out by numerically maximizing the log of the likelihood function in equation (23). In the present case Ordinary Least Squares is used to derive initial

---

\(^7\) This is only an analogy. The variance of the error term does not exist.

\(^6\) A $g$ truncated mean retains $100g$\% of the data. Thus a .25 truncated mean is an average of the central 25$\%$ of the data.

\(^9\) Initial analysis on ISEQ completed to date. Further analysis will follow.
values for the regression parameters. An $\alpha$-stable distribution was fitted to the residuals of this regression using the Mathematica (Wolfram (2003)) $\alpha$-stable density functions in Rimmer (2005). The resulting estimates values of $\alpha$, $\beta$ and $\gamma$ were used as initial values for these parameters. Standard errors of the estimates were estimated by the square root of the diagonal elements of the inverse of the matrix of second derivatives of the loglikelihood function.

The Ordinary Least Squares regression estimates and $\alpha$-stable parameters for the residuals in the day of week effect regression for the ISEQ are given in table (3). The Monday coefficient is the smallest but it is not significantly smaller. Table (4) gives corresponding maximum likelihood estimates of the day of week effects and parameters of the $\alpha$-stable distribution of the residuals. The Monday effect has vanished and the coefficients are now almost the same$^{10}$. The estimates are considerably more precise than the OLS.

[To be completed]

- Comparison of OLS and Stable estimates for other indices
- Has weekday effect changed — split sample in two
- Use of recursive process to speed regression
  1. OLS regression
  2. Estimate $\alpha$-stable parameters of residuals
  3. Re-estimate regression coefficients using equation (27) and the most recent $\alpha$-stable parameters. This will involve another level of recursion as the weights depend on the residuals.
  4. Estimate residuals and estimate $\alpha$-stable parameters of residuals. If these have converged stop. If convergence no reached repeat from step (3)]

$^{10}$The likelihood ratio test of this restriction has a value of .46 and is distributes as $\chi^2$ with 4 degrees of freedom
Table 3: OLS estimates and $\alpha$-stable parameters for week-day effect in ISEQ

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>$t$-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>0.0390</td>
<td>0.0307</td>
<td>1.2699</td>
<td>0.2042</td>
</tr>
<tr>
<td>Tuesday</td>
<td>0.0503</td>
<td>0.0307</td>
<td>1.6379</td>
<td>0.1015</td>
</tr>
<tr>
<td>Wednesday</td>
<td>0.0460</td>
<td>0.0307</td>
<td>1.4980</td>
<td>0.1342</td>
</tr>
<tr>
<td>Thursday</td>
<td>0.0633</td>
<td>0.0307</td>
<td>2.0605</td>
<td>0.0394</td>
</tr>
<tr>
<td>Friday</td>
<td>0.0588</td>
<td>0.0307</td>
<td>1.9138</td>
<td>0.0557</td>
</tr>
</tbody>
</table>

Mean residuals: $1.619 \times 10^{-17}$
S.D. of residuals: 0.9341
Skewness: $-0.3606$
Kurtosis: 5.3777
Jarque-Bera: 5669.45

$\alpha$-stable parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.6390</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$-0.0624$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5013</td>
</tr>
</tbody>
</table>

Table 4: Maximum Likelihood estimates of regression and $\alpha$-stable parameters for week-day effect in ISEQ

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>$t$-statistic</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td>0.0500</td>
<td>0.0250</td>
<td>2.0000</td>
<td>0.0455</td>
</tr>
<tr>
<td>Tuesday</td>
<td>0.0503</td>
<td>0.0247</td>
<td>2.0364</td>
<td>0.0417</td>
</tr>
<tr>
<td>Wednesday</td>
<td>0.0494</td>
<td>0.0258</td>
<td>1.9147</td>
<td>0.0555</td>
</tr>
<tr>
<td>Thursday</td>
<td>0.0690</td>
<td>0.0248</td>
<td>2.7823</td>
<td>0.0054</td>
</tr>
<tr>
<td>Friday</td>
<td>0.0522</td>
<td>0.0248</td>
<td>2.1048</td>
<td>0.0353</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>1.6392</td>
<td>0.0216</td>
<td>75.8888</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$-0.0627$</td>
<td>0.0383</td>
<td>1.6371</td>
<td>0.1016</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5014</td>
<td>0.0071</td>
<td>70.6197</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
5 $\alpha$-Stable Distributions and Value at Risk

Value-at-Risk (VaR) is a commonly used measure of the risk of an investment or of a portfolio. It is often used both in internal risk management and to satisfy the demands of central banks and other regulatory bodies. A $p\%$ VaR is the lower limit on the proportion of a portfolio that can be lost $p\%$ of the time. If $V_p$ is the A $p\%$ VaR then

$$\text{Prob} \{ \text{loss} \geq V_p \} = p \quad (28)$$

Thus if the daily return on the portfolio is normally distributed with an expected value of 0.005% and a standard deviation of 0.010 one would expect to lose

- more than 0.0114% 5% of the time
- more than 0.0183% 1% of the time
- more than 0.0259% 0.1% of the time

The daily VaR of the portfolio is then 0.0114%, 0.0183% and 0.0259% at the 5%, 1% and 0.1% levels respectively. A properly implemented VaR should encompass market, operational and funding risk. It should also facilitate control of operation as a dealer operating outside his limits should effect the calculated VaR if his dealings are recorded by the system. Frain and Meegan (1996) contains an account of the concepts and analytics of Value-at-Risk. For more details see Dowd (1998, 2002) and Jorion (2000).

Table 5 gives estimates of VaR for the six index portfolios at 5%, 1%, 0.1% and 0.01% for returns with normal and $\alpha$-stable distributions. Thus for the ISEQ, assuming normality, these results imply

- a one day loss of more than 1.47% or more once every 20 days or each month.
- a one day loss of more than 2.10% or more once every 100 days or approximately every 5 months.
- a one day loss of more than 2.80% or more once every 1000 days or approximately every 4 years.
- a one day loss of more than 3.36% or more once every 10000 days or 38 years approximately.

Rather than 5%, 1%, 0.1% and 0.01% the relative frequency of these events in the data set are 4.46%, 1.95%, 0.91% and 0.47%. The estimates for the 5% and 1% levels are not unreasonable but the sample frequency in the extreme tail is 47 times the predicted frequency. Using the estimated stable distribution the relative frequency of these classes

$^{11}$VaR data in table 5 have been rounded to three significant figures
Table 5: Comparison of Value-at-Risk under Normal and Stable Distributions

<table>
<thead>
<tr>
<th></th>
<th>Value-at-Risk</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>5.0%</td>
<td>1.0%</td>
<td>0.1%</td>
</tr>
<tr>
<td>ISEQ</td>
<td>Normal</td>
<td>0.0147</td>
<td>0.0210</td>
<td>0.0280</td>
</tr>
<tr>
<td></td>
<td>Stable</td>
<td>0.0132</td>
<td>0.0287</td>
<td>0.1047</td>
</tr>
<tr>
<td>CAC40</td>
<td>Normal</td>
<td>0.0204</td>
<td>0.0289</td>
<td>0.0383</td>
</tr>
<tr>
<td></td>
<td>Stable</td>
<td>0.0193</td>
<td>0.0383</td>
<td>0.1267</td>
</tr>
<tr>
<td>DAX30</td>
<td>Normal</td>
<td>0.0185</td>
<td>0.0261</td>
<td>0.0346</td>
</tr>
<tr>
<td></td>
<td>Stable</td>
<td>0.0165</td>
<td>0.0335</td>
<td>0.1147</td>
</tr>
<tr>
<td>FTSE100</td>
<td>Normal</td>
<td>0.0164</td>
<td>0.0232</td>
<td>0.0309</td>
</tr>
<tr>
<td></td>
<td>Stable</td>
<td>0.0149</td>
<td>0.0294</td>
<td>0.1820</td>
</tr>
<tr>
<td>DJC</td>
<td>Normal</td>
<td>0.0161</td>
<td>0.0228</td>
<td>0.0303</td>
</tr>
<tr>
<td></td>
<td>Stable</td>
<td>0.0137</td>
<td>0.0282</td>
<td>0.0982</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>Normal</td>
<td>0.0156</td>
<td>0.0221</td>
<td>0.0294</td>
</tr>
<tr>
<td></td>
<td>Stable</td>
<td>0.0132</td>
<td>0.0279</td>
<td>0.0994</td>
</tr>
</tbody>
</table>

are 5.87%, 1.83%, 0.72% and 0.41%. This is in much closer agreement with the observed frequency.

Assuming a stable distribution the corresponding results are

- a one day loss\(^{12}\) of more than 1.32% or more once every 20 days or each month.
- a one day loss of more than 2.87% or more once every 100 days or approximately every 5 months.
- a one day loss of more than 10.47% or more once every 1000 days or approximately every 4 years.
- a one day loss of more than 36.00% or more once every 10000 days or 38 years approximately.

The relative frequency of these events in the 4622 sample observations are 5.58%, 0.82%, 0.00% and 0.00% respectively. Similar results hold for the other equity indices.

These results confirm that if one is interested in VaR at a 5% or a 1% level the Normal approximation provides a approximation. In the cases considered here, on average, when

\(^{12}\)VaR data in table 5 have been rounded to three significant figures
an $\alpha$-stable distribution is appropriate the normal approximation overestimates the 5% VaR by about 11% and underestimates the 1% by 27%. It is indeed likely that the VaR estimates from Normal and Stable distributions cross between these two quantiles.

The Normal approximation provides an extremely poor estimate of the tail of the distribution of portfolio returns. The .1% and .01% VaR are underestimated on average by factors of 3.8 and 9.3 respectively. As Extreme loss is an important constituent of risk the use of a Normal distribution gives a large underestimate of risk. Clearly, if one is calculating a 0.1% VaR, is interested in estimating the probability of an extreme loss or one has a long horizon one should take the possibility of a $\alpha$-stable distribution into account.

[ETL (Expected Tail Loss) Dowd (2002) ]
6 Option Valuation when Stock Returns have Stable Distributions
Summary and Conclusions

This paper is an examination of the application of the family of $\alpha$-stable distributions to daily returns on six total return equity indices (ISEQ, FTSE100, DAX40, CAC30, S&P500 and the Dow Jones Composite Total Returns (DJC)).

An $\alpha$-stable distribution is a generalization of the normal distribution. It allows data to be more concentrated at the mean (high peaks) and to have a greater number of extreme values (heavy tails) than a normal distribution would predict. While the normal distribution is symmetric an $\alpha$ stable may be skewed. The normal distribution is determined by two parameters, a location parameter $\mu$ and a spread parameter $\sigma$. An $\alpha$-stable process depends on four parameters. The $\alpha$ parameter determines both the height of the peak of the distribution and the weight of the tails. A $\beta$ parameter determines the skewness of the data ($\beta = 0$ implies no skewness). As for the normal distribution the remaining two parameters determine the location and spread of the data. When $\alpha = 2$ and $\beta = 0$ the stable distribution becomes a normal distribution. Thus the normal distribution is a member of the family of $\alpha$-stable distributions.

$\alpha$-stable distributions have been well known to mathematicians since at least the 1930’s. Some of their properties make them attractive for modeling returns but implementing such procedures requires a lot of computer power. Section 2 of the paper outlines the basic probability theory underlying these processes and explains their relevance to finance.

Early support for the use of $\alpha$-stable processes in economics and finance came from the writings of Mandelbrot and Fama during the 60’s and 70’s. They found that many asset returns had features typical of these processes. After an initial period of research, interest in stable processes appeared to decrease. While computation problems were probably the main cause of this decline, a further contributing factor were the major breakthroughs in finance, achieved at that time. These were largely based on the assumption of an underlying normal distribution. The success of this work using the normal distribution can be gauged by the fact that Nobel prizes have since been awarded to Markovich, Millar, Sharpe, Merton and Scholes for their work on portfolio allocation, Capital Asset Pricing model, Option Pricing and other contributions to the theory of investment. These developments have formed the major constituents of the research agenda in quantitative finance ever since. An erroneous opinion circulating at that time was that the acceptance of $\alpha$-stable distributions would invalidate not only this work but most econometric work completed up to that time. It is only in more recent years that advances in computing facilities have facilitated increased interest in $\alpha$-stable processes and this trend is likely to continue.

One can give many instances of the failure of the normal distribution to account for extreme values. For example, There have been 35 falls greater than 6% in the daily Dow Jones Industrial Average since it was first calculated in 1896, almost 110 years ago. If the returns on the index are normally distributed one would expect that 35 falls of this
The six total return indices considered here are available for shorter periods only but show similar discrepancies in the numbers of large falls in the indices. For example the daily FTSE100 total returns index which is available from 31 December 1985 shows 7 falls greater than 5% in the period to September 2005. Assuming a normal distribution one would expect 7 such falls to occur every 124,000 years. The daily ISEQ total returns index shows 6 falls in the period from January 1989 to September 2005. The normality assumption would imply an expected period of 12,000 years. The distribution of increases show similar discrepancies between the empirical distribution and the normal distribution. Section 3 of the paper addresses this problem. It compares the fit of the normal and $\alpha$-stable distributions to the six total return indices. Apart from one case the fit to the $\alpha$-stable distribution is good. All tests of the normal hypothesis reject that hypothesis. Graphical analysis highlights problems in the fit at the center of the normal distribution as well as in the tails.

Section 4 deals with linear regression when the residuals have an $\alpha$-stable distribution. Standard OLS and maximum likelihood estimators are not the same in this case. The properties of both estimators are outlined and compared. The likelihood function must be maximized numerically but it will provide a better estimate. It is shown that it is a form of robust estimator in that it gives less weight to extreme observations. An existing maximum likelihood method for regression estimates with symmetric $\alpha$-stable residuals is extended to a general process. OLS and maximum likelihood of the weekday effect in ISEQ returns are estimated. The first uses OLS and the second maximum likelihood. While the results are similar the maximum likelihood estimators are more precise. Further analysis will look at the weekday effect in the remaining series and also give separate estimates for the first and second half of each sample.

If return distributions have an $\alpha$-stable distribution, parametric estimates of Value-at-Risk, based on a normal distribution may underestimate the risk to the returns on a portfolio. Section 5 estimates the amount of the understatement of risk involved. These results are based on basic Value-at-Risk (VaR$^{13}$) calculations. Assuming that returns follow an $\alpha$-stable process the 5% (VaR) is over estimated by on average 11% when a normal distribution is used. The 1% VaR is underestimated on average by 27%. However the 0.1% VaR is underestimated by a factor of more than 3. Thus using a normal distribution gives rise to a gross underestimation of the 0.1% VaR measure of risk. This result has implications for portfolio managers in investment companies, bank supervisors and for the stability management function in Central Banks.

In summary for all six total returns indices examined

- The normal distribution is a very poor fit to the empirical distribution of the

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$^{13}$The $p\%$ VaR is measured by the $p\%$ quantile of the return distribution. If $V_p$ is the $p\%$ VaR

\[ P[\text{loss} \geq V_p] = p\% \]
returns on all six indices

- The $\alpha$-stable distribution provides a good fit
- When residuals are $\alpha$-stable, a maximum likelihood estimator has the characteristics of a robust estimator and gives more precise estimates of the regression coefficients.
- Risk as measured by parametric VaR estimates, which assume a normal distribution, underestimates the probability of a large loss.

The $\alpha$-stable family of distributions is a valuable resource in Finance particularly when extreme events are being considered. These techniques should be seen as an extension to and not as a replacement for existing methods based on the normal distribution.
Figure 1: Normal, $\alpha$-Stable and Cauchy Distributions

Figure 2: Tails of Normal, $\alpha$-Stable and Cauchy Distributions
Figure 3: \( \alpha \)-Stable Distribution, \( \alpha = 1.5 \), \( \beta \) various

Figure 4: Comparison of Data, Stable and Normal Distributions
Figure 5: Normal QQ Plot (ISEQ returns) with 95% Limits
Figure 6: Normal QQ Plot (FTSE100 returns) with 95% Limits
Figure 7: Normal QQ Plot (CAC40, DAX30, Dow Composite and S&P100 returns) with 95% Limits
Figure 8: Stable QQ Plot (ISEQ returns) with 95% Limits
Figure 9: Stable QQ Plot (FTSE100 returns) with 95% Limits
Figure 10: Stable QQ Plot (CAC40, DAX30, Dow Composite and S&P100 returns) with 95% Limits
Figure 11: Comparison of implied weights in GLS equivalent of Maximum Likelihood estimates of regression coefficient when residuals are distributed as symmetric $\alpha$ stable variates

Figure 12: Comparison of implied weights in GLS equivalent of Maximum Likelihood estimates of regression coefficient when residuals are distributed as skewed $\alpha$ stable variables with $\beta = -0.1$
References


