Integration and Applications

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EC2040 Topic 5 - Integration and Applications

- **Reading**

1. Chapter 14 of CW
2. Chapters 19 and 20 of PR

- **Plan**

1. Indefinite, definite, and improper integrals
2. Integration (and rules of)
3. Applications

Introduction

- Integration is useful in a number of ways in economics:

1. Microeconomics: consumer surplus; i.e. the difference between what a consumer is willing to pay and what they actually pay.
2. Macroeconomics: stock (e.g., capital) and flow variable (e.g., investment).
3. Finance: net present value of dividend (stock price)

- All of these examples can be thought of as 'areas under curves'.
  - Consumer surplus refers to the demand curve of a consumer; price and quantity space.
  - The macro and finance examples are graphs with time on the horizontal axis.

Areas and Integrals

- Let a and b denote real numbers, where $a < b$.
- Let $f(x)$ be a continuous function (we will focus on functions with one variable).
- What is the area bounded by the curve $y = f(x)$, the vertical lines $x = a$ and $y = b$ and the x-axis?
- What we can do - as a first approximation - is cut the area into rectangles of equal width, where the top right-hand corner touches the curve $y = f(x)$.
- [Diagram]

Areas and Integrals

- Say we split the area into 10 rectangles, where $x_0 = a$ and $x_{10} = b$. The sum of the rectangle areas is the,

$$(x_1 - x_0)\, f(x_1) + (x_2 - x_1)\, f(x_2) + \ldots + (x_{10} - x_9)\, f(x_{10})$$
$$= \sum_{i=1}^{10} = (x_i - x_{i-1})\, f(x_i)$$

- This method leads to some errors, basically, over or underestimation of the area. We can reduce these errors by creating many sub-intervals.
- The sum of the areas of the rectangles tends to a limit as the length of the subintervals tends to zero. This limit gives the area.

$$\int_a^b f(x)dx$$

- This is also called *definite integral*.

Some Simple Examples

- Consider the simplest function, a constant. That is, $y = f(x) = C$. Graphing this is (x, y) space gives a straight line [diagram]. The are between a and b is then height times width. That is $(C - 0) \times (b - a)$.
- However, we can also show,

$$\int_a^b C dx = C \times (b - a)$$

- Likewise, consider the 45 degree line [diagram]. That is $y = f(x) = x$. This involves finding the area of a rectangle $((a - 0) \times (b - a))$ and a triangle $(\frac{1}{2}(b - a)^2)$.
- However, we can also show,

$$\int_a^b x dx = ab - a^2 + \frac{1}{2}(a^2 - 2ab + b^2) = \frac{1}{2}(b^2 - a^2)$$

Integration and Differentiation

- Another useful way of thinking about integration is that it is the reverse process of differentiation.
- The integral or the *antiderivative* of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.
- The function $F(x)$ is referred to as the *indefinite integral*.
- Note that the derivatives of $F(x)$ and $F(x) + C$ (C is a constant) are the same. Hence, the indefinite integral of a function $f(x)$ is only specified up to a constant.

$$\int f(x)dx = F(x) + C$$

- In words, "the integral of $f(x)$ with respect to x is $F(x)$ plus a constant."

The Definite Integral Revisited

- Again, let $f(x)$ be a continuous function on the interval $[a, b]$. Suppose that the function F is an antiderivative of f.
- The difference $F(b) - F(a)$ is then referred to as the *definite integral of f over $[a, b]$*. Why is $F(b) - F(a) = \int_a^b f(x)dx$?
- If $F(x)$ is an antiderivative of f then $G(x) = F(x) + C$ is also an antiderivative for any constant C. However, the value of the definite integral does not depend on the choice of the antiderivative, so,

$$G(b) - G(a) = F(b) + c - [F(a) + c] = F(b) - F(a)$$

- In practical terms, we can then just ignore the constant term when evaluating definite integrals.

Improper Integral

- Sometimes we need to take integrals when the interval is not bounded. Examples:

1. Evaluating the present value of an 'infinite' stream of benefits of a financial asset.
2. Evaluating the consumer surplus of a constant elasticity demand function $q = ap^{-\epsilon}$ (why? it doesn't hit the axis).

- In this case, we have, for example,

$$\int_{x_0}^{\infty} f(x)dx$$

- To motivate the analysis of such integrals, we'll use some examples, later on.

Properties of the Integral

- Since the indefinite integral is the "reverse" of differentiation, we can use the properties of the derivative to derive the following regarding indefinite integrals.

1. $\int af(x)dx = a\int f(x)dx$ [multiple]
2. $\int [f(x) + g(x)]\, dx = \int f(x)dx + \int g(x)dx$ [sum]
3. $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$ [power rule]
4. $\int \frac{1}{x}dx = \ln x + C$ [log rule]
5. $\int e^x dx = e^x + C$ [exponential rule]
6. $\int e^{f(x)} f'(x)dx = e^{f(x)} + C$,
7. $\int [f(x)]^n f'(x)dx = \frac{1}{n+1}[f(x)]^{n+1} + C$ if $n \neq -1$,
8. $\int \frac{1}{f(x)} f'(x)dx = \ln f(x) + C$.

Example: $\int (x^3 + x^2) dx$

- By the second property [sum], $\int (x^3 + x^2) dx = \int x^3 dx + \int x^2 dx$.
- By the third property

$$\int x^3 dx = x^4/4 + C_1 \text{ and } \int x^2 dx = x^3/3 + C_2$$

  where C_1 and C_2 are some constants.
- Since C_1 and C_2 are constants, we can combine them into one and write,

$$\int (x^3 + x^2) dx = x^4/4 + x^3/3 + C$$

- In general, when computing indefinite integrals which involve computing several integrals, we do all the integrals and place a constant at the end.

Example: $\int \left[e^x + \frac{x^2+2x+2}{x^2+2}\right] dx$

- Using the second property [sum], the integral is $\int e^x dx + \int 1 dx + \int \frac{2x}{x^2+2} dx$.
- We know, $\int e^x dx + \int 1 dx = e^x + x + C$
- Property 8 states, $\int \frac{1}{f(x)} f'(x) dx = \ln f(x) + C$ and note the derivative of $x^2 + 2$ is $2x$, i.e., $f(x) = x^2 + 2$.
- We conclude, $\int \frac{2x}{x^2+2} dx = \ln(x^2 + 2) + C$, and,

$$\int \left[e^x + \frac{x^2 + 2x + 2}{x^2 + 2}\right] dx = e^x + x + \ln(x^2 + 2) + C$$

- This looks bad to begin with, but in the end, it turns out to be easy to work with.

Another Example: $\int \left[\frac{(x-2)^2}{x^{0.5}}\right] dx$:

- Expanding the expression inside the integral sign, we have,

$$\int \frac{(x-2)^2}{x^{0.5}} dx = \int \frac{x^2 - 4x + 4}{\sqrt{x}} dx = \int (x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}) dx$$

- Using Property 2,

$$\int \frac{(x-2)^2}{x^{0.5}} dx = \int x^{\frac{3}{2}} dx - \int 4x^{\frac{1}{2}} dx + \int 4x^{-\frac{1}{2}} dx$$

- From Property 3, we get,

$$\int \frac{(x-2)^2}{\sqrt{x}} = \frac{2}{5}x^{\frac{5}{2}} - \frac{8}{3}x^{\frac{3}{2}} + 8x^{\frac{1}{2}} + C$$

Integration by Substitution and Parts

- There are two more powerful rules for integration. They are related to the chain rule and product rules for differentiation.

1. Integration by substitution. This technique operates through a "change of variable" which converts an intractable integral into a form where it can be solved.
2. Integration by parts. This is direct consequence of the product rule. The equivalent expression is,

$$\int u' v dx = uv - \int uv' dx$$

Integration by Substitution, Example: $\int (x^2+10)^{50} 2x dx$

- Examples are the easiest way to understand the substitution rule. Consider expanding $(x^2+10)^{50}$ - this will yield 51 terms which we can then individually integrate.
- However, define a new variable $z = x^2 + 10$. Totally differentiating this gives $dz = 2xdx$.
- Now substitute for $x^2 + 10$ and $2xdx$ to get,

$$\int (x^2+10)^{50} 2xdx = \int z^{50} dz$$

- This latter integral is now easily evaluated using Property 3: $\int z^{50} dz = z^{51}/51 + C$.
- Substitute back for z to get

$$\int (x^2+10)^{50} 2xdx = z^{51}/51 + C = (x^2+10)^{51}/51 + C$$

Integration by Substitution, Example: $\int x\sqrt{1+x}dx$

- Define $u = \sqrt{1+x}$. Taking the square on both sides gives $u^2 = 1 + x$; taking the total differential of this gives $2udu = dx$.
- Now substitute (note that since $u^2 = 1 + x$, we have $x = u^2 - 1$) and we get

$$\int x\sqrt{1+x}dx = \int (u^2-1)u(2udu) = \int 2u^2(u^2-1)du$$
$$= \int 2u^4du - \int 2u^2du = (2/5)u^5 - (2/3)u^3 + C$$

- Now substitute back for u to get,

$$\int x\sqrt{1+x}dx = (2/5)(1+x)^{5/2} - (2/3)(1+x)^{3/2} + C$$

- Differentiate the expression back to confirm that you do get $x\sqrt{1+x}$.

Final Example: $\int \frac{x^3}{(1+x^2)^3} dx$

- Let $u = 1 + x^2$ so that $du = 2xdx$ and $x^2 = u - 1$.
- Without substitution, we can write the above integral as $\int \frac{x^3}{(1+x^2)^3} dx = \int \frac{1}{2} \frac{x^2}{(1+x^2)^3} 2xdx$.
- Now use our substitution.

$$\int \frac{1}{2} \frac{x^2}{(1+x^2)^3} 2xdx = \int \frac{1}{2} \frac{(u-1)}{u^3} du$$
$$= \int \frac{1}{2} \left[(u-1) u^{-3} \right] du = -\frac{1}{2u} + \frac{1}{4u^2} + C$$

- Substituting for u, the integral works out to, $-\frac{1}{2(1+x^2)} + \frac{1}{4(1+x^2)^2} + C$.
- This is a little more tricky. **Figuring out what we substitute in to begin with is the important part.**

Integration by Parts

- Integration by Parts is a consequence of the product rule for differentiation. Recall, that is,

$$(uv)' = u'v + uv'$$

- Integrating both sides of the above expression gives

$$\int (uv)' dx = \int u'v dx + \int uv' dx$$

- Since $\int (uv)' dx = uv$ by definition, we have,

$$\int u'v dx = uv - \int uv' dx$$

- **The first term on the RHS is the product of the integral of u and v and the second term is the integral of a product function which consists of the integral of u and the derivative of v.**

Integration by Parts, Example: $\int x^2 e^x dx$

- Try substitution. It is not useful.
- Instead, use integration by parts and treat x^2 as the function v and e^x as the function u'. We have,

$$\int u'v dx = uv - \int uv' dx \Leftrightarrow$$
$$\int (e^x)(x^2) dx = e^x(x^2) - \int e^x(2x) dx$$

- This doesn't really solve the problem, but we can apply integration by parts (again!) to the second term on the right hand side with e^x as the function u' (as before) and now $2x$ as the function v.

$$\int u'v dx = uv - \int uv' dx \Leftrightarrow$$
$$\int e^x(2x) dx = e^x(2x) - \int e^x(2) dx = 2xe^x - 2e^x$$

Integration by Parts, Example: $\int x^2 e^x dx$

- What does all this substitution get us?
- We need to combine both results.

$$\int (e^x)(x^2)dx = e^x(x^2) - \underbrace{\int e^x(2x)dx}_{=2xe^x - 2e^x}$$

- Adding a constant gives the final result.

$$\int x^2 e^x dx = x^2 e^x - (2xe^x - 2e^x) + C = x^2 e^x - 2xe^x + 2e^x + C$$

- Again, we have integrated a difficult looking function quiet easily.

Integration by Parts, Extended Example (I)

- Consider the more difficult function, $\int x \ln(x+2)dx$.
- Write $x \ln(x+2)$ as $1 \bullet [x \ln(x+2)]$, and choose $u' = 1, v = x \ln(x+2)$.
- We then have,

$$\int u'v dx = uv - \int uv' dx$$

$$\int 1 \bullet [x \ln(x+2)] dx = x[x \ln(x+2)] - \int \underbrace{x\left[\ln(x+2) + \frac{x}{x+2}\right]}_{\text{use the chain rule}} dx$$

Integration by Parts, Extended Example (II)

- By the sum of terms,

$$\underbrace{\int x\ln(x+2)dx}_{\text{once}} = x^2\ln(x+2) - \underbrace{\int x\ln(x+2)dx}_{\text{twice}} - \int \frac{x^2}{x+2}dx$$

$$\Rightarrow 2\int x\ln(x+2)dx = x^2\ln(x+2) - \int \frac{x^2}{x+2}dx$$

- Finally,

$$\int x\ln(x+2)dx = \frac{1}{2}x^2\ln(x+2) - \frac{1}{2}\int \frac{x^2}{x+2}dx$$

Integration by Parts, Extended Example (III)

- We are now left with the task of finding $\int \frac{x^2}{x+2} dx$.
- Write x^2 as $x^2 - 4 + 4$ and note that, $(x^2 - 4) = (x-2)(x+2)$.
- Hence,

$$\int \frac{x^2}{x+2} dx = \underbrace{\int \frac{(x-2)(x+2)}{(x+2)} dx}_{=\int (x-2) dx} + \int \frac{4}{x+2} dx$$

- Now use substitution to conclude that $\int \frac{4}{x+2} dx = 4 \ln(x+2)$ while $\int (x-2) dx = (x^2/2) - 2x$.
- Combining everything and adding a constant, it follows that,

$$\int x \ln(x+2) dx = \frac{1}{2} x^2 \ln(x+2) - \frac{1}{2} \int \frac{x^2}{x+2} dx \Rightarrow$$

$$\int x \ln(x+2) dx = \frac{1}{2} x^2 \ln(x+2) - \frac{1}{2} \left[\frac{x^2}{2} - 2x + 4 \ln(x+2) \right] + C$$

Applications of the Definite Integral

- So far we have not specified the range over which we have integrated.
- Using the definite integral is very useful in economics. For example:

1. The area under the demand curve between two prices p_0 and p_1 corresponds to the change in consumer surplus.
2. The area under the supply curve between two prices p_0 and p_1 corresponds to the change in producer surplus.
3. In finance, the present value of an asset can be approximated as a definite integral.

Some Examples of The Definite Integral

- Suppose we want to evaluate a relatively simple function, $\int_1^5 3x^2 dx$. This is simply $x^3 + C$.
- We know that the area we are interested in is between 1 and 5 on the horizontal axis. Thus we have (it is also clear now why the constant is not relevant), $\left| x^3 + C \right|_1^5 = 125 - 1$.
- Earlier, we integrated $\int x \ln(x+2) dx$ by parts. We want to evaluate this between 0 and 1. We have,

$$\left| \frac{1}{2}x^2 \ln(x+2) - \frac{1}{2}\left[\frac{x^2}{2} - 2x + 4\ln(x+2)\right] + C \right|_0^1$$
$$= \frac{1}{2}1^2 \ln(1+2) - \frac{1}{2}\left[\frac{1^2}{2} - 2 + 4\ln(1+2)\right] - \left\{ -\frac{1}{2}\left[4\ln(2)\right] \right\}$$
$$= \frac{1}{2}\ln(3) - \frac{1}{2}\left[\frac{1}{2} - 2 + 4\ln(3)\right] + \frac{1}{2}\left[4\ln(2)\right]$$

Examples of Integration in Economics

- The idea is, armed with all this knowledge, we can do some economics.
- We look at the following in particular.

1. Consumer's surplus, CES demand functions, and the improper integral
2. Producer's surplus and shut-down prices
3. The net present value of an asset (improper integral again)
4. Growth model (dynamics and integrating over time)

Example I: Consumer Surplus

- Demand curves are just downward sloped curves in price-quantity space. If prices change, consumer welfare changes. We can reformulate this point and ask: How much would a consumer be willing to pay if the price changed from p_0 to p_1, $p_0 > p_1$.
- **Direct gain to price change**: $(p_0 - p_1)\, q_0$. That is, the direct saving on total expenditure based on the original amount, q_0.
- **Indirect gain to price change**: If prices fall a consumer can purchase more of the good. She pays $p(q_0)$ for the first extra Δq. Then, $p(q_0 + \Delta q)$ for the following Δq, $p(q_0 + 2\Delta q)$ for the following Δq, and so on.
- **Net gain to price change**: $\int_{q_0}^{q_1} p(q)\, dq - [p_1(q_1 - q_0)]$. **Total consumer surplus**: $\int_{p_1}^{p_0} q(p)\, dp$.

CES Demand Functions

- To work all of this out we need to specify a demand function. We usually start out by specifying a linear demand function as this looks nice; say, $q = 5 - 3p$. However, this isn't really so useful as the point elasticity is not a constant (check: $(dq/dp)(p/q) = -3[p/(5-3p)]$).
- Consider the following demand function: $q = 30p^{-2}$. This looks bad, but turns out to be useful. Clearly, $dq/dp < 0$. However, $(dq/dp)(p/q) = -60p^{-3}(p/30p^{-2}) = -2$ is a constant. That is, at any point on the demand curve the elasticity of substitution is constant.
- In general, we want to find the consumer surplus of the demand function $q = p^{-\epsilon}$ at $p = p_0$.

CES Demand Functions and Consumer Surplus

- We now run into more problems. It is clear that this function never 'touches' (it asymptotes to) the p axis. The linear demand function didn't suffer with this problem. Thus we now need to evaluate an improper integral.
- There are formal definitions for improper integrals, but we'll use our example to make the point.
- Consumer Surplus can be computed as,

$$\int_{p_0}^{\infty} p^{-\epsilon} dp = \lim_{a\to\infty} \int_{p_0}^{a} p^{-\epsilon} dp = \lim_{a\to\infty} \frac{1}{1-\epsilon}\left[a^{1-\epsilon} - p_0^{1-\epsilon}\right]$$

- Note that the limit exists only when $\epsilon > 1$. In this case, the first term goes to zero as a increases, and so we have,

$$CS = -\frac{1}{1-\epsilon} p_0^{1-\epsilon}$$

Specific Example of CES Demand

- Suppose we have the following demand function, $q = 30p^{-2}$. We know the elasticity is 2. Also suppose that the price is, $p_0 = 2$. This leads to the following:

$$CS\left(p_0 = 2\right) = \int_2^{\infty} 30p^{-2}dp = \lim_{a\to\infty}\int_2^{a} 30p^{-2}dp$$
$$= 30\lim_{a\to\infty}\left|-1/p\right|_2^a = 30\lim_{a\to\infty}\left|0-(-1/2)\right|_2^a = 15$$

- Now consider the alternative CES function, $q = 5 - p^{1/3}$. This looks similar, but presents less of a problem as it clearly touches both the p and q axis. If $p = 0$ then $q = 5$. If $q = 0$ then $p = 5^3 = 125$. Now suppose $p_0 = 2$. We then have the following:

$$CS\left(p_0 = 4\right) = \int_5^{125}\left(5 - p^{1/3}\right)dp = \int_5^{125} 5dp - \int_5^{125} p^{1/3}dp$$
$$= \left|5p - \frac{3}{4}p^{4/3}\right|_5^{125} = 605 - 464 = 141$$

Example II: Marginal Cost and Shut-Down Prices

- Suppose we want to understand how the producer surplus of a firm changes as the price of it's product changes (we will assume perfect competitive where price is given).
- Also suppose the only information we have is the firms marginal cost (that is, how costly it is to produce an additional unit).
- If we had the total cost function we could differentiate it and find the marginal cost function. In that case we can do the opposite. Integrate the marginal cost function and find total costs.
- We find the following.

$$\int MC(q)\,dq = VC(q) + \underbrace{const}_{\text{fixed costs}} = TC(q)$$

- Why is fixed cost$=$ *const*? If we set $q = 0$, we get, $TC(0) = const$.

Shut-Down Prices

- We need to be careful with this analysis. The profit maximizing condition for perfectly competitive firms is price equal to marginal costs; that is, $p = MC(q)$.
- However, firms will only produce if $p > \min AVC(q)$.
- If $p > \min AVC(q)$ is violated, the firm will shut down (that is, produce zero). Using the indefinite integral allows us to calculate $AVC(q)$ and we can therefore work out the shut down price of a firm. That is, we can find out the specific p where the firm chooses $q = 0$?
- To work through all of this we need to assume some more about $MC(q)$.

Shut-Down Prices for a Specific Case

- Assume the following marginal cost function: $MC(q) = 3q^2 + 4q + 2$. In this case, $\int (3q^2 + 4q + 2)\, dq$ implies,

$$TC(q) = q^3 + 2q^2 + 2q + \text{fixed costs}$$

- So, given marginal costs, $VC(q) = q^3 + 2q^2 + 2q$. We also know, $VC(q) = AVC(q)/q = q^2 + 2q + 2$.
- Two properties of $AVC(q)$ are crucial. First, $dAVC(q)/dq = 2q + 2 \geq 0$. Second, $AVC(0) = 2$.
- Given this information we know that a perfectly competitive firm will not produce at a price less than 2. That is, $p = 2$ is the shut-down price.
- For a price greater than or equal to 2, we have $p = MC(q) = 3q^2 + 4q + 2$. The next question is what q will we get for $p > 2$? To find that, we need to solve a quadratic equation.

Solving for Quantities at a Given Price

- Suppose that we want to evaluate producer surplus at $p = 9$. We know $q > 0$ at this price. Specifically, $9 = 3q^2 + 4q + 2$. That is, $3q^2 + 4q - 7 = 0$. However, mathematical, there are two q's that satisfy this condition. Economically, there will only be one. We can see this by applying the quadratic formula.
- Suppose we have $aq^2 + bq + c = 0$. The two solutions are: $(q_1, q_2) = \left[-b \pm \left(b^2 - 4ac\right)^{0.5}\right] / 2a$.
- In our example, $a = 3$, $b = 4$, $c = -7$. That is:

$$q_1 = \frac{-b + \left(b^2 - 4ac\right)^{0.5}}{2a} = \frac{-4 + \left(16 - 4 \cdot 3 \cdot (-7)\right)^{0.5}}{6} = 1$$
$$q_2 = \frac{-b - \left(b^2 - 4ac\right)^{0.5}}{2a} = \frac{-4 - (100)^{0.5}}{6} = -\frac{7}{3}$$

- Clearly, $q_2 < 0$ cannot be the answer. So, we conclude at $p = 2$, $q = 0$ and $p = 9$, $q = 1$.

Calculating Producer Surplus

- With all this information we can now find producer surplus. To do that, we evaluate a definite integral between 0 and 1. 0 relates to the shut down price and 1 relates to $p = 9$, which we have chosen to consider.

$$\int_0^1 [p - MC(q)]\, dq = \int_0^1 \left(9 - 3q^2 - 4q - 2\right) dq$$
$$\left| -q^3 - 2q^2 + 7q \right|_0^1 = -1 - 2 + 7 = 4$$

- We can therefore also ask what happens to producer surplus as price changes. All we need to do is repeat the above analysis for a different price. For example (try this at home), $p = 41$ implies $(q_1, q_2) = (3, -13/3)$. We now evaluate, $\int_0^3 [p - MC(q)]\, dq$ and compare this to $\int_0^1 [p - MC(q)]\, dq$.

Example III: Present Value of an Asset

- Suppose that an asset pays b every year from now. Since a Euro today is not the same as having it a year from now, we discount future benefits. If the discount rate is r, then the benefit b received T years into the future is worth $\frac{b}{(1+r)^T}$ in today's terms.
- Thus, the present value of an asset paying b every year into the future is

$$PV = \frac{b}{(1+r)^0} + \frac{b}{(1+r)^1} + \ldots = \sum_{t=0}^{\infty} \frac{b}{(1+r)^t}$$

- When time becomes 'continuous' it can be shown that the present value of an asset paying an amount at a time T into the future is be^{-rT}. In this case, the present value of the asset is

$$PV = \int_0^{\infty} be^{-rt} dt = \lim_{a \to \infty} -\frac{b}{r}\left[e^{-ra} - 1\right] = \frac{b}{r}$$

- This follows because e^{-ra} goes to zero as a becomes very large.

Example IV: The Domar Growth Model

- Investment is a flow and capital is a stock variable. They have the following relationship: $I(t) = \frac{dK(t)}{dt}$.
- In the Domar Growth model, the rate of change of investment has two effects:

1. Multiplied effect on aggregate demand $\Rightarrow \frac{dY(t)}{dt} = \frac{dI(t)}{dt}\frac{1}{s}$, where s is the marginal propensity to save.
2. An effect on productive capacity.

- Skipping some of the details (see CW), we find the following.

$$\frac{dI(t)}{dt}\frac{1}{I(t)} = \rho s$$

where ρ is the capacity-capital ratio.

Solving the Model

- We can 'solve' this model using the techniques/concepts developed above.
- Integrating both sides implies,

$$\int \frac{dI(t)}{dt}\frac{1}{I(t)}dt = \int \frac{1}{I(t)}dI(t) = \ln|I(t)| + c_1$$
$$\text{and} \int \rho s dt = \rho s t + c_2$$

- Together, we have,

$$\ln|I(t)| = \rho s t + C$$
$$\Leftrightarrow \exp(\ln|I(t)|) = \exp(\rho s t + C)$$
$$I(t) = A\exp(\rho s t) \text{ where } A \equiv \exp(C)$$

Implications of the Domar Model

- At the beginning of time (i.e., $t = 0$), the rate of investment is $I(0) = A$ as $\exp(0) = 1$ (recall, $\ln(1) = 0$).
- Now we can determine the time path of investment from time zero to time t as,

$$I(t) = I(0)\exp(\rho s t)$$

- Thus, to maintain a balance between capacity and demand over time the rate of investment must grow at the exponential rate of ρs.
- Higher capacity-capital ratio or marginal propensity to save requires a higher growth rate.