Optimal and Strategic Terms of Mergers under Two-Source Uncertainty*

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Abstract

This paper presents a continuous time real options model of mergers between two firms experiencing different, but correlated shocks to their profitability. It is assumed that mergers do not just lead to efficiency gains, but are also an act of diversification, leading to a lower volatility for the shocks to the merged firm. Due to the latter assumption the region where a merger is optimal is a bounded interval and not a half-space as in standard real options models. It is shown that if firms can compete in merger activity the option value of mergers vanishes completely. This can lead to substantial differences in the probability of mergers occurring in both scenarios.

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1 Introduction

In the standard literature on real options the cash-flows accruing from an investment project are assumed to be perfectly correlated to one underlying source of uncertainty. Typically, a unique investment threshold is derived above (below) which investment (scrapping) is optimal. In recent years, however, some papers have extended this approach to include multiple sources of uncertainty. Notably, Morellec and Zhdanov (2005) study mergers where the profit flows of both firms are different, but correlated. They derive a unique threshold for the ratio of both profit streams above which a merger or acquisition is optimal. These results are then used to explain excess returns on shares prior to merger and acquisition (M&A) announcements. In this paper we study a model of mergers and acquisitions where the optimal investment region turns out to be a bounded set instead of a half-space.

The real options approach views the possibility of mergers and acquisitions as an option comparable to an American call option. The underlying asset in this paper is a firm's discounted future profit stream. This profit stream is assumed to be subject to risk. We consider a two-factor model with two expected profit maximising firms, which face different, but correlated, risk. A crucial assumption in this paper is that a merger is seen partly as an act of diversification, which leads to less overall volatility of the merged firm than the sum of the volatilities of the firms before a merger. Hence, apart from a (possible) efficiency effect, there is also a diversification effect. This is the main difference with Morellec and Zhdanov (2005), where the volatility of the merged firm is a weighted sum of the volatilities of its constituent firms.

Two scenarios related to M&A activity are analysed. In the first scenario it is assumed that only one firm can engage in M&A activity. That is, a merger is always the result of one firm taking over the other firm. A takeover takes place as soon as one firm makes a bid on the other firm which the other firm does not reject. It is shown that the option value of M&A activity in this case is positive. We find that M&A activity can take place both during economic expansions and contractions. The most important factor in determining the optimality of a merger or takeover decision is the relative profit of a firm vis à vis the other firm.

However, unlike Morellec and Zhdanov (2005), the optimal region can be reached from above as well as from below in our model. If the ratio of profits[1] is very low, a merger is not profitable for the acquirer, since the stand-alone profits will outweigh the (possible) synergy effect of the merger. Alternatively, if the ratio is very high, a merger is not profitable since the shareholders of the target have to be compensated substantially for their firm. This implies that, in the absence of speculative bubbles,

[1] The profit of the acquirer (target) in the numerator (denominator).

the option value of an acquisition converges to zero both if the profit ratio tends to zero and if it goes to infinity. This happens because the net present value of a merger is not linear in the ratio of profits, but a polynomial. This, in turns, is a consequence of the diversification effect of mergers, which reduces the probability of favourable future profits.

In the second scenario we consider the case where both firms can engage in M&A activity. A takeover, again, takes place if one firm makes an offer that is accepted by the other firm. A merger takes place if both firms make a bid simultaneously. In this case the profit shares are determined by a Nash bargaining procedure. We show that it is optimal for one firm to make a bid if and only if it is optimal for the other firm to make a bid as well. This result holds regardless of the relative size of the firms. Consequently, in equilibrium both firms will always simultaneously make a bid, i.e. (hostile) takeovers will never take place in equilibrium. Furthermore, the option value of M&A activity vanishes completely in case both firms can engage in it. A numerical analysis shows, in addition, that the probability of investment can be substantially lower in the optimal scenario when compared with the strategic scenario. Given that in most real markets the second scenario is more appropriate, this leads to the prediction that the likelihood of a sub-optimal merger decision is substantial. It is important to note that this sub-optimal decision is in fact maximising shareholder value. Again, this is a different result than obtained in Morellec and Zhdanov (2005), where the optimal investment region is different for both firms and M&A activity depends on the relative strength of firms.

There is a substantial literature that analyses mergers and takeovers as real options. Margrabe (1978) is the first to recognise that takeovers are exchange options in a model with exogenous timing. The first real options analysis of mergers is Lambrecht (2004), who studies a model where the underlying source of uncertainty is the same for both firms. As a result he finds that mergers only take place during times of economic expansion. Our paper is most closely related to Morellec and Zhdanov (2005) who extend Lambrecht (2004) to a situation where the firms face different, but correlated, sources of uncertainty. The present paper extends Morellec and Zhdanov (2005) to a situation where a merger is not a pure exchange option due to a diversification effect. This has important consequences for the the analysis of mergers and, for example, the probability with which mergers take place.

The paper is organised as follows. In Section 2 the case where only one firm can engage in M&A activity is analysed. In Section 3 we analyse the case where both firms can engage in M&A activity. Finally, Section 4 discusses the results.

2 The Optimal Timing of Acquisitions

Consider two firms, indexed by $i \in \{1, 2\}$, which operate in separate, but related markets. Let $\mathcal{P} = (\Omega, \mathcal{F}, P)$ be a filtered probability space. The profit flow of firm i at time $t \in [0, \infty)$, denoted by π_t^i, consists of a deterministic part, denoted by $D_i > 0$, and a stochastic component, denoted by X_{it}, which is adapted to $\mathcal{P}$. The deterministic component can be thought of as resulting from competition in the product market. The stochastic shock is assumed to be multiplicative, that is,

$$\pi_{it} = X_{it} D_i.$$

The stochastic shock follows a geometric Brownian motion with trend μ_i and volatility σ_i, i.e.

$$dX_{it} = \mu_i X_{it} dt + \sigma_i X_{it} dW_{it}, \tag{1}$$

where W_i is a Wiener process. The instantaneous correlation between W_1 and W_2 equals $\rho \in (-1, 1)$. It is assumed that the discount rate for both firms is constant and equal to $r > 0$. Furthermore, in order for the problem to have a finite solution it is assumed that $\mu_i < r$, for $i \in \{1, 2\}$.

Suppose that firm 1 is the larger firm, which has an option to take over firm 2, leading to a combined deterministic profit flow $D_m > 0$. For simplicity, it is assumed that the takeover process does not involve sunk costs.[2] After the takeover it is assumed that the weight of market 1 for the new firm equals $\gamma \in (0, 1)$. Since firm 1 is assumed to be the larger firm it is logical to take $\gamma \geq \frac{1}{2}$. Instead of an arithmetic average of the two sources of uncertainty as in Schleifer and Vishny (2003) and Morellec and Zhdanov (2005), we take a geometric average of the shocks.[3] So, the stochastic shock that the merged firm faces at time t, denoted by Y_t, equals

$$Y_t = X_{1t}^{\gamma} X_{2t}^{1-\gamma}.$$

For further reference, the process $(Z_t)_{0 \leq t < \infty}$ is defined, where, for all $t \geq 0$, $Z_t = \frac{X_{1t}}{X_{2t}}$.

The following lemma states that Y and Z follow geometric Brownian motions. Its proof is an elementary application of Ito's lemma and is, therefore, omitted.

[2] Sunk costs of takeovers can be thought of to comprise, for example, the legal costs of the takeover (including the costs incurred for getting formal approval by competition authorities), the costs of restructuring the two organisations to facilitate the takeover, etc.

[3] The rationale for this functional form is best understood by considering the deterministic case, i.e. $\sigma_1 = \sigma_2 = 0$. Then it holds that $X_t^i = e^{\mu_i t}$ for $i = 1, 2$. Hence, the growth rate of the profit of firm i equals μ_i. The growth rate of the merged firm should then equal $\gamma\mu_1 + (1-\gamma)\mu_2$. In other words, $Y = e^{\gamma\mu_1 + (1-\gamma)\mu_2} = X_1^{\gamma} X_2^{1-\gamma}$.

Lemma 1 *There exist Wiener processes* $\big(W_t^Y\big)_{0\leq t<\infty}$ *and* $\big(W_t^Z\big)_{0\leq t<\infty}$*, such that the processes* $(Y_t)_{0\leq t<\infty}$ *and* $(Z_t)_{0\leq t<\infty}$ *are adapted to* $\mathcal{P}$ *and follow geometric Brownian motions, equal to*

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y, \tag{2}$$

$$dZ_t = \mu_Z Z_t dt + \sigma_Z Z_t dW_t^Z, \tag{3}$$

respectively, where

$$\mu_Y = \gamma\mu_1 + (1-\gamma)\mu_2 - \frac{1}{2}\gamma(1-\gamma)\left((\sigma_1-\sigma_2)^2 + 2\sigma_1\sigma_2(1-\rho)\right), \tag{4}$$

$$\sigma_Y^2 = (\gamma\sigma_1 + (1-\gamma)\sigma_2)^2 - 2\gamma(1-\gamma)\sigma_1\sigma_2(1-\rho), \tag{5}$$

$$\mu_Z = \mu_1 - \mu_2 + \sigma_2(\sigma_2 - \sigma_1\rho), \tag{6}$$

$$\sigma_Z^2 = (\sigma_1+\sigma_2)^2 - 2\sigma_1\sigma_2(1+\rho). \tag{7}$$

Note that $\mu_Y < r$. Furthermore, it holds that $\mu_Y < \gamma\mu_1 + (1-\gamma)\mu_2$. Hence, the trend of the uncertainty faced by the merged firm is lower than the weighted average of the trends of the separate firms. This is offset, though, by a smaller volatility, since $\sigma_Y^2 < (\gamma\sigma_1 + (1-\gamma)\sigma_2)^2$. Hence, a takeover can be seen as an act of diversification, comparable to an investor creating a portfolio with different assets to diversify risk.[4]

It is assumed throughout that each firm maximises expected discounted profits. In complete and efficient markets this represents the market value of the firm. Since we focus on the details of the timing decision, it is assumed that shareholders have perfect information to simplify the exposition found in Morellec and Zhdanov (2005). If the acquirer decides to takeover the target at time t, the value to its shareholders is denoted by $V(X_{1t}, X_{2t})$.

Suppose that firm 1 decides to take over firm 2 at time $\tau \geq 0$. Then firm 1 has to compensate the shareholders of firm 2 for “losing” their firm. The profit stream of the newly formed firm will be YD_m, while the stand-alone profit stream of firm 2 equals X_2D_2. So, the management of firm 1 should offer the shareholders of firm 2 a profit share $s_\tau \in [0,1]$, such that the expected discounted value of the new firm is at least as high as the expected discounted stand-alone value. That is, s_τ should be such that

$$\mathbb{E}\Big(\int_\tau^\infty e^{-rt} s_\tau Y_t D_m dt\Big) \geq \mathbb{E}\Big(\int_\tau^\infty e^{-rt} X_{2t} D_2 dt\Big). \tag{8}$$

Since the management of firm 1 maximises its own market value, (8) holds with equality in an optimum. Standard computations[5] show that $\mathbb{E}(\int_t^\infty e^{-rs} Y_s ds) =$

[4] Note, however, that mergers actually reduce the risk-sharing possibilities of investors. Whether merger synergies offset this loss to shareholders is an open question.

[5] See e.g. Huisman (2001).

$\frac{Y_t}{r-\mu_Y}$. Hence, solving (8) gives

$$\begin{aligned} s_\tau &= \frac{D_2}{D_m}\frac{r-\mu_Y}{r-\mu_2}\frac{X_{2\tau}}{Y_\tau} \\ &= \frac{D_2}{D_m}\frac{r-\mu_Y}{r-\mu_2}\Big(\frac{X_{2\tau}}{X_{1\tau}}\Big)^\gamma. \end{aligned} \tag{9}$$

The expected discounted value of the acquisition at time $t \geq 0$ is, therefore, equal to

$$\begin{aligned} V(X_{1t}, X_{2t}) &= \mathbb{E}\Big(\int_\tau^\infty e^{-rt}(1-s_t)Y_t D_m dt\Big) \\ &= \frac{D_m}{r-\mu_Y}Y_t - \frac{D_2}{r-\mu_2}\Big(\frac{X_{1t}}{X_{2t}}\Big)^{-\gamma}Y_t \\ &= X_{2t}\Big[\frac{D_m}{r-\mu_Y}Z_t^\gamma - \frac{D_2}{r-\mu_2}\Big]. \end{aligned} \tag{10}$$

Let $\mathcal{T}$ denote the set of stopping times for $(X_t)_{t\geq 0}$, where $X_t = (X_{1t}, X_{2t})$, for all $t \geq 0$. The problem for firm 1 is to solve the following optimal stopping problem: Find $G^*(x_1, x_2)$ and $T^* \in \mathcal{T}$ such that

$$\begin{aligned} G^*(x_1, x_2) &= \sup_{T\in\mathcal{T}} \mathbb{E}\Big[\int_0^T e^{-rt}D_1X_{1t}dt + e^{-rT}V(X_{1T}, X_{2T})\Big] \\ &= \mathbb{E}\Big[\int_0^{T^*} e^{-rt}D_1X_{1t}dt + e^{-rT^*}V(X_{1T^*}, X_{2T^*})\Big]. \end{aligned} \tag{11}$$

Proposition 1 *Let β_1 and β_2 be the positive and negative root, respectively, of the quadratic equation*

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_Z^2\beta(\beta-1) + (\mu_1-\mu_2)\beta - (r-\mu_2) = 0.$$

Furthermore, suppose that

$$\gamma\frac{D_m}{r-\mu_Y} > \Big(\frac{D_1}{r-\mu_1}\Big)^\gamma\Big(\frac{\gamma}{1-\gamma}\frac{D_2}{r-\mu_2}\Big)^{1-\gamma}. \tag{12}$$

Then there exist pairs (A_1, Z_1) and (A_2, Z_2) such that the optimal stopping problem (11) is solved by $(G^*(\cdot), T^*)$, where

$$G^*(x_1, x_2) = \begin{cases} x_2\Big(A_1\Big(\frac{x_1}{x_2}\Big)^{\beta_1} + \frac{D_1}{r-\mu_1}\frac{x_1}{x_2}\Big) & \textit{if } 0 \leq \frac{x_1}{x_2} < Z_1 \\ \frac{D_m}{r-\mu_Y}x_1^\gamma x_2^{1-\gamma} - \frac{D_2}{r-\mu_2}x_2 & \textit{if } Z_1 \leq \frac{x_1}{x_2} \leq Z_2 \\ x_2\Big(A_2\Big(\frac{x_1}{x_2}\Big)^{\beta_2} + \frac{D_1}{r-\mu_1}\frac{x_1}{x_2}\Big) & \textit{if } \frac{x_1}{x_2} > Z_2, \end{cases} \tag{13}$$

and

$$T^* = \inf\{t \geq 0 | Z_t \in [Z_1, Z_2]\}. \tag{14}$$

Proof. Instead of the standard method of solving the optimal stopping problem (11) via the Bellman equation (cf. Dixit and Pindyck (1996)) we use the fact that (11) is similar to the Dirichlet problem with free boundary. For details on the mathematical background, see Appendix A or Øksendal (2000, Chapter 10).

The problem (11) is not time-homogeneous. Consider, therefore, the stochastic process $B_t = (s+t, X_{1t}, X_{2t}, P_t)$, defined by

$$dB_t = \begin{bmatrix} 1 \\ \mu_1 X_{1t} \\ \mu_2 X_{2t} \\ e^{-rt} D_1 X_{1t} \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma_1 X_{1t} \\ \sigma_2 X_{2t} \\ 0 \end{bmatrix} dW_t,$$

where W_t is a 4-dimensional Brownian motion. Then

$$G^*(x_1, x_2) = \sup_{T \in \mathcal{T}} \mathbb{E}[P_T + e^{-rT} V(X_{1T}, X_{2T})] = \sup_{T \in \mathcal{T}} \mathbb{E}[G(B_T)], \tag{15}$$

with

$$G(b) = e^{-rs} V(x_1, x_2) + p.$$

The optimal stopping problem (15) is a time-homogeneous optimal stopping problem that is equivalent to (11). Therefore, we can apply Øksendal (2000, Theorem 10.4.1) (see Appendix A) to problem (15).

Pivotal in the proof is the following lemma, the proof of which can be found in Appendix B.

Lemma 2 *If (12) holds, then the following systems of equations permit a solution in* (A, Z)*,*

$$A_1 Z_1^{\beta_1} + \frac{D_1}{r-\mu_1} Z_1 = \frac{D_m}{r-\mu_Y} Z_1^{\gamma} - \frac{D_2}{r-\mu_2} \tag{16}$$

$$A_1 \beta_1 Z_1^{\beta_1 - 1} + \frac{D_1}{r-\mu_1} = \gamma \frac{D_m}{r-\mu_Y} Z_1^{\gamma-1}, \tag{17}$$

and

$$A_2 Z_2^{\beta_2} + \frac{D_1}{r-\mu_1} Z_2 = \frac{D_m}{r-\mu_Y} Z_2^{\gamma} - \frac{D_2}{r-\mu_2} \tag{18}$$

$$A_2 \beta_2 Z_2^{\beta_2 - 1} + \frac{D_1}{r-\mu_1} = \gamma \frac{D_m}{r-\mu_Y} Z_2^{\gamma-1}, \tag{19}$$

where the solutions are such that $A_1 > 0$*,* $A_2 > 0$*, and* $Z_1 < Z_2$*.*

Assume that the continuation region is of the form $D = \{(s, x_1, x_2) | 0 < \frac{x_1}{x_2} < Z_1\} \cup \{(s, x_1, x_2) | \frac{x_1}{x_2} > Z_2\}$, for some $0 < Z_1 < Z_2$. Define $\tau_D := \inf\{t \geq 0 | B_t \notin D\}$

and compute $F(s,x_1,x_2,p) = \mathbb{E}[G(\tau_D)]$ in the following way. From Øksendal (2000, Theorem 9.2.14) it follows that F solves the Dirichlet problem, i.e. it is the bounded solution to the boundary value problem

$$\begin{cases} \mathcal{L}_X F = 0 \quad \text{in } D \\ \lim\limits_{x_1/x_2 \to Z^*} F(s,x_1,x_2) = g(s,Z^*), \end{cases}$$

where $\mathcal{L}_X$ is the partial differential operator,

$$\begin{aligned} \mathcal{L}_{(X_1,X_2)}F =& \frac{\partial F}{\partial s} + \mu_1 X_1 \frac{\partial F}{\partial x_1} + \mu_2 X_2 \frac{\partial F}{\partial x_2} + e^{-rs} x_1 D_1 \frac{\partial F}{\partial p} \\ &+ \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 F}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 F}{\partial x_2^2} + \sigma_1\sigma_2\rho x_1 x_2 \frac{\partial^2 F}{\partial x_1 \partial x_2} = 0. \end{aligned} \tag{20}$$

If we impose that $F(\cdot)$ is of the form

$$F(s,x_1,x_2,p) = e^{-rs} x_2 \varphi(z) + p,$$

with $z = x_1/x_2$, the partial derivatives of $F(\cdot)$ become $\frac{\partial F}{\partial s} = -re^{-rs}x_2\varphi(z)$, $\frac{\partial F}{\partial x_1} = e^{-rs}\varphi'(z)$, $\frac{\partial F}{\partial x_2} = e^{-rs}(\varphi(z) - z\varphi'(z))$, $\frac{\partial^2 F}{\partial x_1^2} = e^{-rs}\varphi''(z)/x_2$, $\frac{\partial^2 F}{\partial x_2^2} = e^{-rs}z^2\varphi''(z)/x_2$, $\frac{\partial^2 F}{\partial x_1 \partial x_2} = -e^{-rs}z\varphi''(z)/x_2$, and $\frac{\partial F}{\partial p} = 1$. Hence, (20) becomes

$$\begin{aligned} \mathcal{L}_{(X_1,X_2)}F = e^{-rs}x_2\Big[&- r\varphi(z) + \mu_1 z\varphi'(z) + \mu_2\big(\varphi(z) + z\varphi'(z)\big) + zD_1 \\ &+ \frac{1}{2}\sigma_1^2 z^2 \varphi''(z) + \frac{1}{2}\sigma_2^2 z^2\varphi''(z) - \sigma_1\sigma_2\rho z^2 \varphi''(z)\Big] = 0 \\ \iff \frac{1}{2}\sigma_Z^2 z^2 \varphi''(z) &+ (\mu_1 - \mu_2) z\varphi'(z) - (r - \mu_2)\varphi(z) + zD_1 = 0. \end{aligned} \tag{21}$$

The partial differential equation (21) has the general solution

$$\varphi(z) = A_1 z^{\beta_1} + A_2 z^{\beta_2} + \frac{D_1}{r - \mu_1} z,$$

where β_1 and β_2 solve $\mathcal{Q}(\beta) = 0$, and A_1 and A_2 are constants. The boundedness condition on the solution implies that it should hold that $\lim_{z \downarrow 0} \varphi(z) = 0$ and $\lim_{z \to \infty} \varphi(z) = 0$.[6] It is easy to see that $\beta_1 > 1$ and $\beta_2 < 0$. Therefore, it should hold that $A_2 = 0$ on $[0, \bar{Z})$ and $A_1 = 0$ on $(\bar{Z}, \infty)$.

If (A_1, Z_1) and (A_2, Z_2) satisfy the boundary conditions (16) and (18), respectively, a candidate solution for (11) is obtained:

$$F(t,x_1,x_2) = \begin{cases} e^{-rt}\left(x_2 A_1 \left(\frac{x_1}{x_2}\right)^{\beta_1} + \frac{D_1}{r-\mu_1}x_1\right) & \text{if } 0 < \frac{x_1}{x_2} < Z_1 \\ e^{-rt}\left(\frac{D_m}{r-\mu_Y} x_1^{\gamma} x_2^{1-\gamma} - \frac{D_2}{r-\mu_2}x_2\right) & \text{if } Z_1 \le \frac{x_1}{x_2} \le Z_2 \\ e^{-rt}\left(x_2 A_2 \left(\frac{x_1}{x_2}\right)^{\beta_2} + \frac{D_1}{r-\mu_1}x_1\right) & \text{if } \frac{x_1}{x_2} > Z_2. \end{cases}$$

[6] These conditions also rule out the existence of speculative bubbles. See Dixit and Pindyck (1996, Section 6.1.C).

If (A_1, Z_1) and (A_2, Z_2) in addition satisfy the smooth pasting conditions (17) and (19), respectively, it holds that $\varphi \in C^1$.

It is easy to see that B_t spends 0 time on ∂D a.s., that ∂D is a Lipschitz surface, that $\varphi \in C^2(\mathbb{R}\backslash\partial D)$ with locally bounded second order derivatives near ∂D, that $\tau_D < \infty$ a.s., and that the family $\{\varphi(Y_\tau)|\tau < \tau_D\}$ is uniformly integrable for all $y \in \mathbb{R}$. By construction, it holds that $\mathcal{L}_X\varphi = 0$ on D. Furthermore, from

$$\begin{aligned}
\mathcal{L}_{(X_1,X_2)}F =& e^{-rs}\frac{D_m}{r-\mu_Y}x_1^\gamma x_2^{1-\gamma}\Big(-r+\gamma\mu_1+(1-\gamma)\mu_2 \\
&-\frac{1}{2}\gamma(1-\gamma)(\sigma_1^2+\sigma_2^2+2\sigma_1\sigma_2\rho)\Big) \\
=& -e^{-rs}x_2\frac{D_2}{r-\mu_2}(r+\mu_2) \\
<& -e^{-rs}\frac{D_m}{r-\mu_Y}x_1^\gamma x_2^{1-\gamma}(r-\mu_Y) \\
<& 0,
\end{aligned}$$

it follows that $\mathcal{L}_X\varphi \leq 0$, for $\frac{x_1}{x_2} \notin D$. Finally, we can see that $\varphi(\cdot) \geq V(\cdot)$, which follows immediately from the following lemma and is a direct corollary to Lemma 2.

Lemma 3 *Define* $f_1(z) = A_1 z^{\beta_1} + \frac{D_1}{r-\mu_1}z$, $f_2(z) = A_2 z^{\beta_2} + \frac{D_1}{r-\mu_1}z$, *and* $g(z) = \frac{D_m}{r-\mu_Y}z^\gamma - \frac{D_2}{r-\mu_2}$. *It holds that, if* $0 \leq x_1/x_2 < Z_1$, *then* $f_1(z) > g(z)$. *If* $x_1/x_2 > Z_2$, *then* $f_2(z) > g(z)$.

Since all conditions of Theorem A.1 are satisfied, the pair $(F(\cdot), \tau_D)$ solves the optimal stopping problem (11). □

From Proposition 1 it becomes clear that not the *absolute* profitability of firms is important, but *relative* profitability. It, therefore, does not follow directly that takeovers take place during economic booms. This results from the fact that $[Z_1, Z_2]$ is reached either from below on $[0, Z_1)$, or from above on (Z_2, ∞). On $[0, Z_1)$ a takeover can take place either if firm 1 experiences a sharper upswing than firm 2, or a slower downturn. In both cases Z is increasing. On (Z_2, ∞) a takeover can take place if firm 1 experiences a sharper downturn than firm 2 or a slower upswing. In both cases Z is decreasing. Note that Lambrecht (2004) concludes unequivocally that mergers only take place during economic booms. This happens because in his model both firms are subject to the same random process from the outset.

An important result of this model is that takeovers can only be optimal if the synergies are high enough. To see this consider a case where $\mu_1 = \mu_2 \equiv \mu$, $\gamma = \frac{D_1}{D_1+D_2}$, and $D_m = (1+\alpha)(D_1 + D_2)$, where α is a synergy parameter. These synergies can arise from increased production efficiency or a decrease in competition, or a combination of both. It is easy to see that, in this case, (12) holds iff $\alpha >$

$\underline{\alpha} \equiv \frac{r-\mu_Y}{r-\mu} - 1$.[7] Note that the lower bound $\underline{\alpha} \geq 0$ holds for all feasible parameter configurations.

Furthermore, $\underline{\alpha}$ is decreasing in ρ, with $\underline{\alpha} = 0$ for $\rho = 1$. In other words, the higher the degree of diversification (i.e. the smaller ρ) the higher the minimally required synergies. The intuition behind this result is that due to risk-neutrality the firm does not care about volatility. The higher ρ, the lower the volatility σ_Y and the lower the trend μ_Y. In order to offset the reduction in trend and, hence, expected discounted profits, the higher the synergies need to be. That is, the diversification argument is not important for a risk neutral firm.

Finally, $\underline{\alpha}$ is quadratic in γ. For $\gamma \in \{0, 1\}$, the minimally required synergies equal $\underline{\alpha} = 0$. The threshold $\underline{\alpha}$ is maximal for $\gamma = \frac{1}{2}$. Again, this is due to risk-neutrality. The diversification effect is maximal for $\gamma = \frac{1}{2}$, but investors do not value the diversification. What they do care about is the lower trend resulting from diversification. Therefore, they need to be compensated with a higher synergy effect.

3 The Strategic Timing of Mergers and Acquisitions

In this section, the model from the previous section is extended to a situation where both firms can decide to make an acquisition offer. Throughout, it is assumed that if both firms simultaneously make an offer, a merger is agreed upon. We follow the basic setup for simple timing games as described in Fudenberg and Tirole (1991, Section 4.5).

Each firm has the choice to make an acquisition offer at each point in time t. So, the strategy set for firm i at time t is

$$S_i(t) = \{\text{make offer}, \text{don't make offer}\}.$$

Suppose that at time t, firm 1 makes an acquisition offer to firm 2. In the terminology of timing games this makes firm 1 the "leader". Firm 2 is the "follower" in this case. The payoff to firms 1 and 2 are (cf. (10))

$$L_1(X_{1t}, X_{2t}) = X_{2t}\Big[\frac{D_m}{r-\mu_Y} Z_\tau^\gamma - \frac{D_2}{r-\mu_2}\Big]$$

and

$$F_2(X_{1t}, X_{2t}) = X_{1t}\Big[\frac{D_2}{r-\mu_2} \frac{1}{Z_t}\Big],$$

[7] It holds that if $\alpha = \underline{\alpha}$, then $Z_1 = Z_2 = 1$.

respectively. In the case firm 2 makes an acquisition offer, while firm 1 does not, the payoffs are given by

$$L_2(X_{1t}, X_{2t}) = X_{1t}\Big[\frac{D_m}{r-\mu_Y}\Big(\frac{1}{Z_t}\Big)^{1-\gamma} - \frac{D_1}{r-\mu_1}\Big]$$

and

$$F_1(X_{1t}, X_{2t}) = X_{2t}\Big[\frac{D_1}{r-\mu_1}Z_t\Big],$$

respectively.

If both firms simultaneously make an acquisition offer at time t it is assumed that a merger takes place. The firms use the Nash bargaining solution (Nash (1950)) with disagreement point $d = \Big(\frac{D_1}{r-\mu_1}X_{1t}, \frac{D_2}{r-\mu_2}X_{2t}\Big)$ to determine how to split the value $\frac{D_m}{r-\mu_Y}Y_t$. The bargaining power of firm 1 is assumed to be equal to its relative market power, γ. It is easily shown that this leads to the merger payoffs

$$\begin{aligned} M_1(X_{1t}, X_{2t}) &= \frac{D_m}{r-\mu_Y}Y_t + \frac{1}{2}\Big(\frac{D_1}{r-\mu_1}X_{1t} - \frac{D_2}{r-\mu_2}X_{2t}\Big) \\ &= \gamma L_1(X_{1t}, X_{2t}) + (1-\gamma)F_1(X_{1t}, X_{2t}), \end{aligned}$$

and

$$\begin{aligned} M_2(X_{1t}, X_{2t}) &= \frac{D_m}{r-\mu_Y}Y_t + \frac{1}{2}\Big(\frac{D_2}{r-\mu_2}X_{2t} - \frac{D_1}{r-\mu_1}X_{1t}\Big) \\ &= (1-\gamma)L_2(X_{1t}, X_{2t}) + \gamma F_2(X_{1t}, X_{2t}), \end{aligned}$$

respectively.

The following lemma determines the region where the leader payoff is larger, respectively smaller, for both firms. The proof can be found in Appendix C.

Lemma 4 *Suppose that (12) holds. Then there exists an interval $D_P = [\tilde{Z}_1, \tilde{Z}_2]$, for certain $\tilde{Z}_1$ and $\tilde{Z}_2$, such that*

$$Z \in D_P \iff L_1(Z) \geq F_1(Z) \text{ and } L_2(Z) \geq F_2(Z).$$

Furthermore, it holds that $\tilde{Z}_1 \leq Z_1 \leq Z_2 \leq \tilde{Z}_2$.

Lemma 4 shows that there exist values of Z where both firms want to be the leader. Even stronger: Firm 1 wants to acquire firm 2 if and only if firm 2 wants to acquire firm 1. This result holds *irrespective* of the relative market power parameter γ.

In the region D_P it holds that $L_i(x_1, x_2) \geq M_i(x_1, x_2) \geq F_i(x_1, x_2)$, with strict inequalities in the interior. At each point in time both firms basically play the state game depicted in Figure 1. If $z \notin D_P$, not making an offer is a dominant strategy for

	make offer	don't make offer
make offer	$\Big(M_1(X_{1t}, X_{2t}), M_2(X_{1t}, X_{2t})\Big)$	$\Big(L_1(X_{1t}, X_{2t}), F_2(X_{1t}, X_{2t})\Big)$
don't make offer	$\Big(F_1(X_{1t}, X_{2t}), L_2(X_{1t}, X_{2t})\Big)$	$\Big(F_1(X_{1t}, X_{2t}), F_2(X_{1t}, X_{2t})\Big)$

Figure 1: The state game.

both firms. For $z \in D_P$, making an offer is a (weakly) dominant strategy for both firms. Let $T_P = \inf\{t \geq 0 | Z_t \in D_p\}$. Note that $T_P(\omega) \in \bar{\mathbb{R}}$ for all $\omega \in \Omega$. A strategy for firm i consists of a distribution function $G_i : \mathbb{R}_+ \to [0,1]$, where $G_i(t)$ is the probability that firm i has invested before time t. It is easily seen that a subgame perfect equilibrium (in weakly dominant strategies) is given by

$$G_i(t) = \begin{cases} 0 & \text{if } 0 \leq t < T_P \\ 1 & \text{if } t \geq T_P. \end{cases} \tag{22}$$

From (22) it follows that the option value, which in the one firm case is given by $A_1 Z^{\beta_1}$ or $A_2 Z^{\beta_2}$, completely disappears when both firms can acquire each other. This is contrary to the standard real options literature where competition erodes the option value, albeit it does not vanish completely (cf. Thijssen (2004, Chapter 4)). In the case of M&A activity the option value completely vanishes, because it results from a zero-sum game. In order to acquire the other firm, a firm must pay the other firm's shareholders its expected discounted stand-alone value. Furthermore, it forgoes its own expected stand-alone value. The expected discounted value of the merged firm offsets this stand-alone value if and only if this holds for the other firm as well. Therefore, it is optimal for firm 1 to acquire firm 2 if and only if it is optimal for firm 2 to acquire firm 1. That is, we should only observe friendly mergers in a market. A hostile takeover is (in this framework) always a dominated strategy.

The fact that (22) holds irrespective of the market power parameter γ is caused by the assumption that if both firms choose to make a bid in the game depicted in Figure 1, the division of the profits is given by the asymmetric Nash bargaining solution, where the bargaining power of firm 1 equals γ, so that the effect of γ is internalised. Furthermore, the disagreement point is not a credible option in the region D_P, since if firm i is acquired in region D_P it gets exactly its expected discounted stand-alone value, whereas the Nash bargaining solution always gives *at least* this value.

4 Discussion

The analysis from the previous section shows that mergers are more likely in cases where both firms can preempt each other. An interesting question is how much more likely it is that a merger takes place with competition. We consider a situation with $D_2 = 100$, $\mu_1 = \mu_2 = 0.03$, $r = 0.1$, $\sigma_1 = 0.1$, $\sigma_2 = 0.15$, $\rho = 0.8$, and synergies $\alpha = 0.1$. The deterministic profit stream for firm 1 is taken to be $D_1 \in [D_2, 3D_2]$. That is, $\gamma \in [0.5, 0.75]$. The values for Z_1, Z_2, $\tilde{Z}_1$, and $\tilde{Z}_2$ are depicted in Figure 2. As one can see, the bounds for the optimal and strategic timing are relatively close.

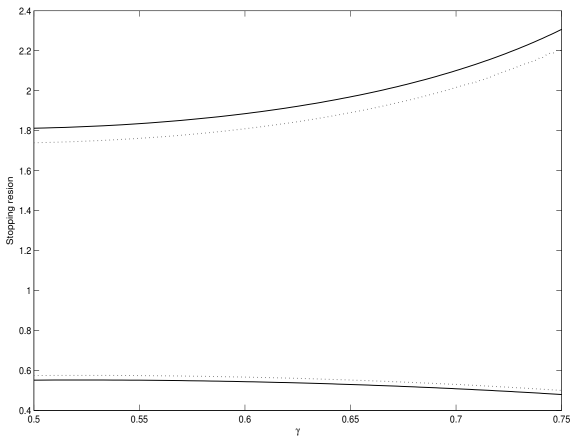

Figure 2: Strategic (solid lines) and optimal (dotted lines) investment regions.

This could give the impression that the effect of competition in mergers is not very big. This would be misleading as follows from Figure 3, which depicts the probability of investment within $T = 50$ periods as a function of γ. The investment region can be reached from below or from above. For $Z_0 < \bar{Z}$ (see Appendix B for a definition of $\bar{Z}$), the probability of investment before time T equals (cf. Harrison (1985))

$$\begin{aligned} \mathbb{P}\big(\sup_{0\le t\le T} Z_t \ge Z \big| Z_0\big) =& \mathcal{N}\left(\frac{-\log(Z/Z_0) + \bar{\mu}T}{\sigma_V\sqrt{T}}\right) \\ &+ \left(\frac{Z}{Z_0}\right)^{\frac{2\bar{\mu}}{\sigma_V^2}} \mathcal{N}\left(\frac{-\log(Z/Z_0) - \bar{\mu}T}{\sigma_V\sqrt{T}}\right), \end{aligned} \tag{23}$$

for $Z \in \{Z_1, \tilde{Z}_1\}$, where $\bar{\mu} = \mu_Z - \frac{1}{2}\sigma_Z^2$. For $Z_0 > \bar{Z}$, the probability of investment before time T is given by

$$\mathbb{P}\big(\inf_{0\le t\le T} Z_t \le Z \big| Z_0\big) = \mathbb{P}\big(\sup_{0\le t\le T} -Z_t \ge -Z \big| Z_0\big),$$

for $Z \in \{Z_2, \tilde{Z}_2\}$. For the computations we have taken $Z_0 = 1/3$ when the optimal region is reached from below, whereas $Z_0 = 3$ is taken when the optimal region is reached from above. As is clear from this figure, a small difference in the bounds can

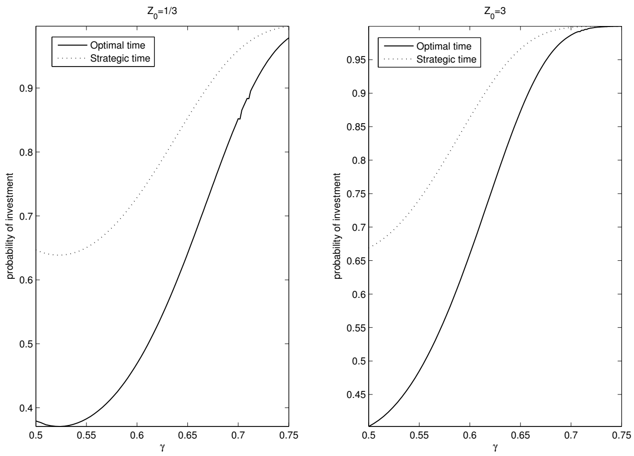

Figure 3: Probability of investment within $T = 50$ periods from below (left-panel) and above (right-panel).

lead to very different probabilities of investment. This happens, because the trend and the volatility of Z are quite small, $\mu_Z = 0.0105$ and $\sigma_Z = 0.0085$, respectively.

Furthermore, it is interesting to see how the probability of investment changes with the initial value Z_0. To analyse this, we take $D_1 = D_2 = 100$ (i.e. $\gamma = 1/2$), $\alpha = 0.08$, and a horizon of $T = 20$. The other parameter values are taken to be the same as before. This leads to a situation where $\{\tilde{Z}_1, Z_1, Z_2, \tilde{Z}_2\} = \{0.637, 0.667, 1.507, 1.570\}$. Note that μ_Z and σ_Z are the same as before. The resulting probabilities are depicted in Figure 4. Again, one observes the substantial difference in the probability of investment at the optimal and the strategic time. Note, furthermore, that not taking into account the upper bound on the optimal investment region would lead to the erroneous conclusion that the probability of investment equals one for all values of Z_0 in the right-panel.

Finally, the asymmetry between firms, as measured by γ, also has an influence on the option value of the merger in the optimal timing scenario. Note that β_1 and β_2 are independent of γ. It can be seen from (4) that μ_Y and, therefore, the net present value of the merger is decreasing for $\gamma \in [0.5, 1]$. As a result one expects that the option value increases with γ, which is confirmed by Figure 5, which plots the constants A_1 and A_2 as defined by (16)–(19) for the same scenario as described above. This increase in the option value, which has a negative impact on the probability of a merger taking place, is due to the diversification effect. Risk-neutrality is a crucial assumption here. It is assumed that investors only care about expected payoffs. Therefore, the only effect in the valuation of the merger is the

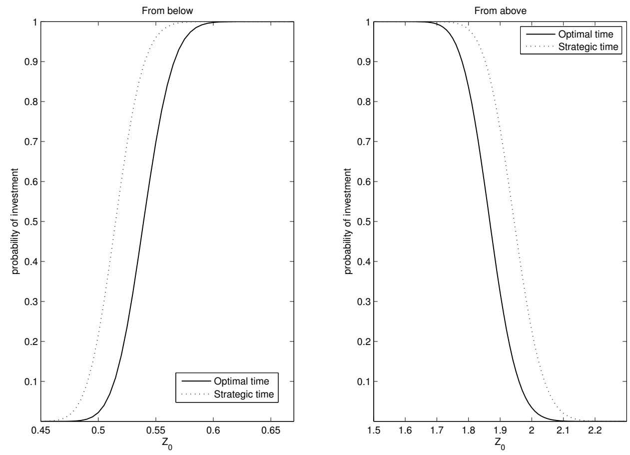

Figure 4: Probability of investment within 20 periods from below (left-panel) and above (right-panel).

negative effect of a decreased trend due to to diversification. It is to be expected that a risk-averse investor would attach positive value to diversification, which might mitigate the effect obtained in this model.[8] On the other hand, one could argue that, by merging, the firms reduce the possibilities of diversification for shareholders and should, therefore, not take risk aversion into consideration. As a price for reducing flexibility for the shareholder they need to establish higher synergies.

To conclude, the main results of this analysis are that, firstly, mergers can take place both during expansions and contractions, due to the boundedness of the optimal investment region and the dependence of the optimal time on the ratio of the firms' shocks. Secondly, the option value of a merger completely vanishes if there is competition for takeovers, due to the threat of preemption. This leads to the prediction that takeovers are likely to take place at suboptimal points in time. Finally, the probability of investment can be substantially different in both scenarios (optimal and strategic timing) even if the two investment regions are relatively close.

Therefore, the likelihood that a merger takes place at a sub-optimal time is substantial in case there is a preemptive threat. This could be considered to hold for most real-world mergers and takeovers. It is important to note that this sub-optimal timing takes place due to maximisation of shareholder value. In the language of modern corporate finance this means that it takes place *in the interest of the shareholder*. Most shareholders will, however, hold a well-diversified portfolio and may,

[8] The shocks X_i, $i = 1, 2$ are often interpreted as being specified under the equivalent martingale measure, which is used in defense of the assumption of risk-neutrality. Note that such an argument cannot be used here.

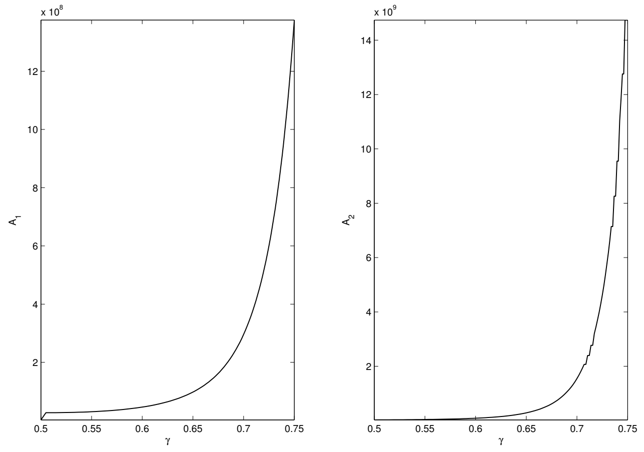

Figure 5: Value of the constants A_1 (left-panel) and A_2 (right-panel).

therefore, hold shares in both firms. As such, a merger reduces the possibilities of risk-spreading for shareholders. In essence, the firms are diversifying for the shareholders, who, consequently, lose the flexibility to dynamically update any portfolio they might have had of the constituent firms. This might have a negative effect on shareholder wealth and might require even higher synergies to offset this loss of flexibility. A thorough analysis of this problem is, however, left for future research.

Appendix

A Optimal Stopping Theory

Let $(X_t)_{t\geq 0}$ be an Ito diffusion on a domain $V \subset \mathbb{R}^n$, defined by

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t,$$

with $dB_i dB_j = \rho_{ij}dt$ and $\rho_{ii} = 1$, for all $i = 1, \ldots, n$. A time-homogenous optimal stopping problem on $(X_t)_{0\leq t<\infty}$, with *reward function* $g : V \to \mathbb{R}_+$ and *instantaneous reward function* $f : V \to \mathbb{R}$, is of the form: Find (g^*, τ^*) such that

$$g^*(x) = \sup_{\tau} \mathbb{E}\Big[\int_0^{\tau} f(X_t)dt + g(X_\tau)\Big] = \mathbb{E}\Big[\int_0^{\tau^*} f(X_t)dt + g(X_{\tau^*})\Big], \qquad \text{(A.1)}$$

the supremum being taken over all stopping times τ for $(X_t)_{0\leq t<\infty}$. Define

$$T = \sup\{t > 0 | X_t \notin V\}.$$

Furthermore, define the the *partial differential operator* $\mathcal{L}_X$,

$$\mathcal{L}_X = \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} (\sigma\sigma')_{ij}(y)\rho_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Consider a function $\varphi : \bar{V} \to \mathbb{R}$ and the set $D = \{x \in V | \varphi(x) > g(x)\}$. The following theorem is from Øksendal (2000, p. 213).

Theorem A.1 (Variational inequalities for optimal stopping) *If the following conditions hold:*

1. $\varphi \in C^1(V) \cap C(\bar{V})$;
2. $\varphi \geq g$ *on* V *and* $\varphi = g$ *on* ∂V;
3. $\mathbb{E} \int_0^T \mathbb{1}_{\partial D}(X_t) dt = 0$;
4. ∂D *is a Lipschitz surface;*
5. $\varphi \in C^2(V \backslash \partial D)$ *and the second order derivatives of* φ *are locally bounded near* ∂D;
6. $\mathcal{L}_X \varphi + f \leq 0$ *on* $V \backslash \bar{D}$;
7. $\mathcal{L}_X \varphi + f = 0$ *on* D;
8. $\tau_D := \inf\{t > 0 | X_t \notin D\} < \infty$ *a.s.;*
9. *the family* $\{\varphi(X_\tau) | \tau \leq \tau_D\}$ *is uniformly integrable w.r.t. the probability law of* X_t.

Then $g^*(x) = \varphi(x) = \sup_{\tau \leq T} \mathbb{E}\Big[\int_0^\tau f(X_t) dt + g(X_\tau) \Big]$, *and* $\tau^* = \tau_D$, *solve the optimal stopping problem (A.1).*

B Proof of Lemma 2

Define the functions $f_1(z) = A_1 z^{\beta_1}$, $f_2(z) = A_2 z^{\beta_2}$, and $g(z) = \frac{D_m}{r-\mu_Y} z^\gamma - \frac{D_1}{r-\mu_1} z - \frac{D_2}{r-\mu_2}$. Applying the first and second order conditions yields that g has a global maximum at

$$\bar{Z} = \left(\gamma \frac{D_m}{D_1} \frac{r-\mu_1}{r-\mu_Y} \right)^{\frac{1}{1-\gamma}}.$$

Under condition (12) it holds that $g(\bar{Z}) > 0$. Given that $f_1'(z) > 0$ and $f_2'(z) < 0$ this immediately leads to the desired result. See also Figure 6 . □

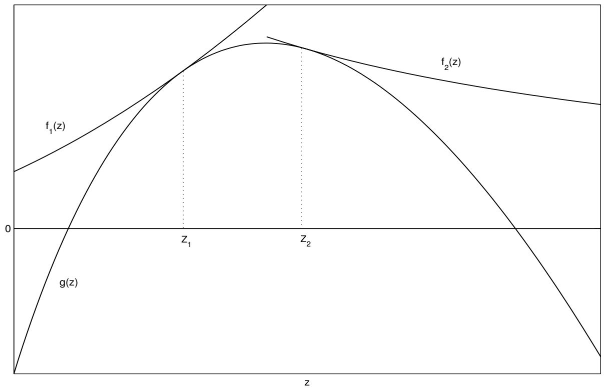

Figure 6: Graph of the functions f_1, f_2, and g.

C Proof of Lemma 4

Note that

$$L_1(x_1,x_2) \geq F_1(x_1,x_2) \iff \frac{D_m}{r-\mu_Y}z^\gamma - \frac{D_2}{r-\mu_2} \geq \frac{D_1}{r-\mu_1}z$$
$$\iff g_1(z) \equiv \frac{D_m}{r-\mu_Y}z^\gamma - \frac{D_1}{r-\mu_1}z - \frac{D_2}{r-\mu_2} \geq 0.$$

from the proof of Lemma 2 we know that under (12), the function $g_1(\cdot)$ has a global maximum at, say, z^*, with $g_1(z^*) > 0$. Since $g_1(\cdot)$ is strictly concave, this implies that there exist $\tilde{Z}_1$ and $\tilde{Z}_2 > \tilde{Z}_1$ such that $L_1(z) \geq F_1(z) \iff \tilde{Z}_1 \leq z \leq \tilde{Z}_2$.

Furthermore, it holds that

$$L_2(x_1,x_2) \geq F_2(x_1,x_2) \iff g_2(z) \equiv \frac{D_m}{r-\mu_Y}(1/z)^{1-\gamma} - \frac{D_2}{r-\mu_2}(1/z) - \frac{D_1}{r-\mu_1} \geq 0.$$

Since $g_2(z) = zg_1(z)$, it holds that $g_2(\cdot)$ has the same zeros as $g_1(\cdot)$ on $\mathbb{R}_{++}$. The second part of the lemma follows immediately from Figure 6. $\square$

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