On General Equilibrium Effects and Contingent Claim Valuation of Financial Assets*

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Abstract

This paper studies the extent of the error that is made in standard contingent claim analysis, which underlies modern asset pricing theories and real option theory within a two-period general equilibrium model with incomplete markets. It is well-known that in mean-variance, or CARA-normal economies the introduction of new assets leaves the prices of existing assets, relative to the bond, unchanged. Simulations show that contingent claim valuation remains a good predictor of a new asset's equilibrium price in CRRA-lognormal economies with habit formation. Present value, however, performs badly. Equilibria are computed via a differentially implementable homotopy.

Keywords: Asset pricing, General equilibrium, Incomplete markets.
JEL codes: D52, G12.

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1 Introduction

The Contingent Claim (CC) approach to valuation of financial assets is central to modern financial economics. The basic idea is to use observed prices to price a new, non-traded asset. This asset is usually taken to be a derivative security (see, for example, Duffie (1996), Musiela and Rutkowski (2005), or Cochrane (2005) for excellent overviews). These ideas have also been applied to capital budgeting in the so-called real option theory (see Dixit and Pindyck (1994) for an overview).

In essence, the CC approach regresses the payoff stream of the asset-to-be-priced (henceforth called the "new asset") on the span of all traded assets (or a subset thereof). This gives the *replicating portfolio*, which is then priced using current asset prices. The payoff stream of the replicating portfolio is often called the *spanning asset*, as it spans the payoffs of the new asset. It is usually assumed that markets are complete, or at least "complete enough", in the sense that the payoff stream of the new asset is in the span of traded assets. This implies that the replicating portfolio is unique. In regression terms, the R^2 of regressing the payoff stream on the market span equals one. In economic terms, the risk in the new asset can be perfectly hedged in the market.

In real-world applications two important issues arise. Firstly, it can be argued that markets are incomplete, i.e. that not all risk in the new asset can be hedged.[1] This implies that, generically, for practical purposes the revenue stream does not lie in the market span and only a partial spanning asset can be obtained. As a result, an error might occur in valuing the new asset. This error is the difference between the actual price of the asset were it traded and the price obtained by CC analysis.

Secondly, the introduction of a new asset might change the prices of all other traded assets as well. There is empirical evidence that the introduction of new batches of options has substantially changed asset prices between 1973 and 1986 (see Conrad (1989) and Detemple and Jorion (1990)). Some theoretical papers have been devoted to this topic. Weil (1992) and Elul (1997) show that the introduction of a new asset permits agents to better share risk. This weakens the need for precautionary savings and, hence, leads to a higher interest rate. This, in turn, reduces the prices of all assets in the economy. Oh (1996) shows that in economies with mean-variance preference, or CARA preferences and normally distributed asset payoffs, the price of any risky asset relative to the riskless bond is unaffected by changes in the market span. In a recent paper, Calvet et al. (2004) show that

[1] Even if markets are, in fact, complete, analysis usually uses only a subset of all traded assets. Valuation then takes place *as if* markets are incomplete. See Ross (2005) for an elaboration of this point.

in a CARA-normal economy with limited participation relative prices are in fact influenced by financial innovation.

In this paper, the question of financial innovation is studied via a simulation study of a two-period general equilibrium model with incomplete markets (GEI) in the spirit of Magill and Quinzii (1996, Chapter 2). We model agents with (identical) CRRA preferences, which exhibit habit formation in the sense of Abel (1990). Agent heterogeneity is introduced via non-identical initial endowments, which consist of labour income and income from initial asset holdings. We model the economy before band after the introduction of the new asset, economies $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively. The equilibrium price of the new asset in $\tilde{\mathcal{E}}$ is then compared with its CC value, which is obtained from $\mathcal{E}$. It is shown that this CC value is a remarkably good predictor of the actual equilibrium price of the new asset. This resembles results presented in Herings and Kubler (2003) who show that CAPM valuation works extremely well in CRRA economies. The result is indeed surprising, given the fact that in our simulations, the replicating portfolio is usually a very poor predictor of the true payoff stream, in the sense of having a low R^2. This results in the present value being a bad predictor for the market value of the new asset. In the simulations, a homotopy technique introduced in Herings and Kubler (2002) is used to compute the equilibria.

The paper is organised as follows. In Section 2, the GEI model is introduced. In Section 3 the investment project is described in detail and in Section 4 the computational study is presented. Finally, Section 5 discusses the simulation results.

2 The GEI Finance Economy

The General Equilibrium model with Incomplete markets (GEI) explicitly includes incomplete financial markets in a general equilibrium framework. In this paper the simplest version is used. It consists of two time periods, $t = 0, 1$, where $t = 0$ denotes the present and $t = 1$ denotes the future. At $t = 0$ the state of nature is known to be $s = 0$. Uncertainty over possible states of nature at $t = 1$ is modelled by a probability space $\mathcal{S} = (S, \mathcal{S}, P)$, where S is assumed to be a finite set indexed (with slight abuse of notation) by $s = 1, \ldots, S$. In the economy there are $H \in \mathbb{N}$ investors, or agents, indexed by $h = 1, \ldots, H$. There is one consumption good, which can be interpreted as income. A consumption plan for investor $h \in \{1, \ldots, H\}$ is a vector $x^h \in \mathbb{R}_+^{S+1}$, where x_s^h gives the consumption level in state $s \in \{0, 1, \ldots, S\}$.[2]

Each investor $h = 1, \ldots, H$, is characterised by a vector of initial endowments,

[2] In general a vector $x \in \mathbb{R}^{S+1}$ is denoted $x = (x_0, \boldsymbol{x_1}) \in \mathbb{R} \times \mathbb{R}^S$ to separate x_0 in period $t = 0$ and $\boldsymbol{x_1} = (x_1, \ldots, x_S)$ in period $t = 1$.

$\omega^h \in \mathbb{R}_+^{S+1}$, and a utility function $u^h : \mathbb{R}_+^{S+1} \to \mathbb{R}$. Denote aggregate initial endowments by $\omega = \sum_{h=1}^{H} \omega^h$. Regarding the initial endowments and utility functions the following assumptions are made.

Assumption 1 *The vector of aggregate initial endowments is strictly positive, i.e.* $\omega \in \mathbb{R}_{++}^{S+1}$.

Assumption 2 *For each investor $h = 1, \ldots, H$, the utility function, u^h, is continuous, strictly monotone and strictly quasi-concave on $\mathbb{R}_+^{S+1}$.*

Assumption 1 ensures that in each period and in each state of nature there is at least one agent who has a positive amount of the consumption good. Assumption 2 ensures that the consumer's demand is a continuous function.

It is assumed that the market for the consumption good is a spot market. The investors can smoothen consumption by trading on the financial market, where $J \in \mathbb{N}$ financial contracts are traded, indexed by $j = 1, \ldots, J$. The future payoffs of the assets are put together in a matrix

$$A = (A^1, \ldots, A^J) \in \mathbb{R}^{S \times J},$$

where A_s^j is the payoff of one unit of asset j in state s. The following assumption is made with respect to the matrix A.

Assumption 3 *There are no redundant assets, i.e. $rank(A) = J$.*

Assumption 3 can be made without loss of generality; if there are redundant assets then a no-arbitrage argument guarantees that its price is uniquely determined by the other assets. Let the market subspace be denoted by $\langle A \rangle = Span(A)$. That is, the market subspace consists of those income streams that can be generated by trading on the financial market. If $S = J$, the market subspace consists of all possible income streams, i.e. markets are complete. If $J < S$ there is idiosyncratic risk and markets are incomplete.

A *GEI economy* is defined as a tuple $\mathcal{E} = \Big((u^h, \omega^h)_{h=1,\ldots,H}, A\Big)$. Given a GEI economy $\mathcal{E}$, investor h can trade assets by buying a portfolio $\theta^h \in \mathbb{R}^J$ given the (row)vector of prices $q = (q_0, \boldsymbol{q_1}) \in \mathbb{R}^{J+1}$, where q_0 is the price for consumption in period $t = 0$ and $\boldsymbol{q_1} = (q_1, \ldots, q_J)$ is the vector of security prices with q_j the price of security j, $j = 1, \ldots, J$.[3] Given a vector of prices $q = (q_0, \boldsymbol{q_1}) \in \mathbb{R}^{J+1}$, the budget set for investor $h = 1, \ldots, H$ is given by

$$B^h(q) = \Big\{x \in \mathbb{R}_+^{S+1} \Big| \exists_{\theta \in \mathbb{R}^J} : q_0(x_0 - \omega_0^h) \leq -\boldsymbol{q_1}\theta, x_{\boldsymbol{1}} - \omega_{\boldsymbol{1}}^h = A\theta \Big\}. \qquad (1)$$

[3]We follow the convention of denoting prices in row vectors and quantities in column vectors.

Given the asset payoff matrix A we will restrict attention to asset prices that generate no arbitrage opportunities, i.e. asset prices q such that there is no portfolio generating a semi-positive income stream. Such asset prices exclude the possibility of "free lunches". The link between utility maximisation follows from the following theorem (cf. Magill and Quinzii (1996)).

Theorem 1 (Fundamental Theorem of Finance) *Let $\mathcal{E}$ be a finance economy satisfying Assumption 2. Then the following conditions are equivalent:*

1. $q \in \mathbb{R}^{J+1}$ *permits no arbitrage opportunities;*
2. $\forall_{h=1,\dots,H} : \arg\max\{u^h(x^h) | x^h \in B^h(q)\} \neq \emptyset$*;*
3. $\exists_{\pi \in \mathbb{R}^S_{++}} : q_{\mathbf{1}} = \pi A$*;*
4. $B^h(q)$ *is compact for all* $h = 1, \dots, H$*.*

The vector $\pi \in \mathbb{R}^S_{++}$ can be interpreted as a vector of state prices. Condition 3 therefore states that a no-arbitrage price for security j equals the present value of security j given the vector of state prices π. As a consequence of this theorem, in the remainder we restrict ourselves to the set of no-arbitrage prices

$$Q = \{q \in \mathbb{R}^{J+1} | q_0 > 0, \exists_{\pi \in \mathbb{R}^S_{++}} : q_{\mathbf{1}} = \pi A\}. \tag{2}$$

An important consequence of Theorem 1 is that in complete markets state prices are uniquely determined. If one normalises state prices on the unit simplex, π can be interpreted as a probability measure. Since no-arbitrage prices are simply the expected value of asset payoffs under π, this probability measure is usually referred to as the *martingale measure*. Furthermore, note that Theorem 1 does not require equilibrium considerations at all. So, under complete markets only Assumption 2 is needed for asset pricing. If markets are incomplete, however, π is not uniquely determined. This is exactly the reason why asset pricing in incomplete markets is conceptually much more difficult.

Under Assumption 2, Theorem 1 shows that the demand function $x^h(q)$, maximising investor h's utility function $u^h(x)$ on $B^h(q)$, is well-defined for all $h = 1, \dots, H$, and all $q \in Q$. It can easily be shown that $x^h(q)$ and the security demand function, $\theta^h(q)$, determined by $A\theta^h(q) = x^h_{\mathbf{1}}(q) - \omega^h_{\mathbf{1}}$, are continuous on Q.

Define the *excess demand* function $f : Q \to \mathbb{R}^{J+1}$ by

$$f(q) = \big(f_0(q), f_{\mathbf{1}}(q)\big) = \Big(\sum_{h=1}^{H}(x^h_0(q) - \omega^h_0), \sum_{h=1}^{H}\theta^h(q) - \Theta\Big),$$

where $\Theta \in \mathbb{R}^J_+$ denotes each asset's net-supply. A *financial market equilibrium* (FME) for a GEI economy $\mathcal{E}$ is a tuple $\big((\bar{x}^h, \bar{\theta}^h)_{h=1,\dots,H}, \bar{q}\big)$ with $\bar{q} \in Q$ such that:

1. $\bar{x}^h \in \arg\max\{u^h(x^h)|x^h \in B^h(\bar{q})\}$ for all $h = 1, \ldots, H$;

2. $A\bar{\theta}^h = \bar{x}^h_{\mathbf{1}} - \omega^h_{\mathbf{1}}$ for all $h = 1, \ldots, H$;

3. $\sum_{h=1}^{H} \bar{\theta}^h = 0$.

It is easy to show that $q \in Q$ is an FME iff $f(q) = 0$. The following result is proved in e.g. Hens (1991) and Talman and Thijssen (2006).

Theorem 2 *Let $\mathcal{E}$ be a GEI economy satisfying Assumptions 1–3. Then there exists $q \in Q$ such that $f(q) = 0$.*

3 The Value of a Financial Innovation

In this section, different methods of valuing an financial innovation are described. Let $\mathcal{S} = (S, \mathcal{S}, P)$ be a (discrete) probability space and let $\mathcal{E} = (u, \omega, A)$ be a two-period GEI economy, with J assets. Suppose that a new asset is introduced with future payoffs $A^{J+1} \in \mathbb{R}^S$. This new asset can be a new financial product which is actually going to be traded on the financial market. It could also represent the risky payoffs of a real investment project of a firm. In this case the asset will not actually be traded, but the firm wishes to value the project taking into account the shareholders' interests.[4]

The present value of the new asset asset is simply

$$PV(A^{J+1}) = \delta \mathbb{E}_P(A^{J+1}), \tag{3}$$

where δ is the discount rate and the expectation is taken under the measure P. If the new asset is an investment project The PV is most closely related to the effect of investment on product markets as it is solely based on expected profits. As such it should incorporate effects of the investment project on market structure and competition. For a publicly listed firm, however, the owners are not just interested in expected profits, but also in higher order moments and covariances between the return on the investment project and returns on other traded financial assets.

In recognition of this observation, contingent claim analysis (CC) assumes that there exists a (unique) replicating portfolio for the return A^{J+1}. Using this portfolio and the (current) asset prices $q \in \mathbb{R}^J$ one can value the asset. However, if financial markets are incomplete, generically, no unique replicating portfolio exists. Following Föllmer and Sondermann (1986) one could use the projection of A^{J+1} on $\langle A \rangle$ instead. Let $\theta_A(A^{J+1})$ denote the (unique) replicating portfolio of $proj_{\langle A \rangle}(A^{J+1})$,

[4] As is the standard assumption concerning investment appraisal in corporate finance textbooks (cf. Brealey and Myers (2003)).

where $proj_{\langle A\rangle}(x)$ denotes the projection in $||\cdot||_2$ of $x \in \mathbb{R}^S$ onto $\langle A\rangle$. The contingent claim value, denoted by $CC(A^{J+1})$, is then

$$CC(A^{J+1}) = q\theta_A(A^{J+1}). \tag{4}$$

Obviously, this procedure is nothing else than running a linear regression with A^{J+1} as dependent variable and the existing assets in A as regressors and is, hence, an attractive procedure for applications. The main difference with PV is that CC does not use an exogenously determined discount factor, but computes an "expected value" using risk adjusted discounting and probabilities derived from market prices. A clear advantage of this procedure is that one does not need to appeal to equilibrium prices. For CC valuation no-arbitrage is the only condition needed. This is yet another reason for the popularity of the CC approach. However, if markets are incomplete, CC asset valuation might lead to structural errors if it holds that $A^{J+1} \notin \langle A\rangle$.

Furthermore, if one computes the value of a new asset according to (4) it is implicitly assumed that the projection of its payoffs carries over to equilibrium prices and that $\langle A\rangle$ would not change were an asset with payoffs A^{J+1} actually traded on the market. However, adding an asset implies the market will be less incomplete and changes $\langle A\rangle$, which has an influence on the equilibrium prices of *all* assets.[5]

Let $\tilde{A} = \begin{bmatrix} A & A^{J+1} \end{bmatrix} \in \mathbb{R}^{S\times J+1}$ be the asset payoff matrix *after* the new asset has been introduced. Note that if initial endowments consist partly of initial asset holdings, changes in asset prices also change initial endowments to, say, $\tilde{\omega}$. That is, after an asset with payoffs A^{J+1} has been introduced, the new GEI economy is $\tilde{\mathcal{E}} = (u, \tilde{\omega}, \tilde{A})$. Let $\tilde{q} \in \mathbb{R}^{J+1}$ be a vector of equilibrium prices in this economy. Then the GEI value of the new asset, denoted by $GEI(A^{J+1})$, is

$$GEI(A^{J+1}) = \tilde{q}_{J+1}. \tag{5}$$

The main difference with CC analysis is that CC values the investment project in the economy $\mathcal{E}$, whereas GEI values the investment in the (possibly hypothetical) economy $\tilde{\mathcal{E}}$. Note that, if the project is perfectly correlated with a convex combination of the J original assets, the GEI and CC values coincide.

So far, only market value has been taken into account. Standard corporate finance textbooks claim that management should maximise *shareholder value* and that it does so by maximising NPV (Brealey and Myers (2003, Chapter 6)). Real option theory has shown that the latter claim is wrong by applying CC analysis with

[5]Oh (1996) shows that in mean-variance and CARA normal economies prices of risky assets relative to the riskless bond remain unchanged.

(usually) risk neutral investors. Still, CC analysis assumes that market value is a good indicator of shareholder wealth. In complete markets with risk neutral investors this is correct. In a GEI setting, however, risk aversion, market incompleteness, and investor heterogeneity may lead to different results. In standard general equilibrium models one often uses the equivalent or compensating variation to measure effects on agents' wealth of price and/or income changes. Since the equivalent variation takes current prices as the starting point this seems the better approach is this setting. The equivalent variation of the new asset A^{J+1}, $EV(A^{J+1})$, is defined as

$$EV(A^{J+1}) = \sum_{h=1}^{H} e^h(q, u^h) - q_{\mathbf{1}}\theta^h_{\mathbf{1}},$$

where for all h, $e^h(q, u^h)$ solves

$$e^h(q, \bar{u}^h) = \min_{\{\theta \in \mathbf{R}^{J+1} | \omega^h + W\theta \geq 0\}} \{q_{\mathbf{1}}\theta | u^h(\omega^h_0 - \tfrac{q_{\mathbf{1}}}{q_0}\theta, \omega^h_{\mathbf{1}} + \tilde{A}\theta) \geq \bar{u}^h\},$$

$W = \begin{bmatrix} -q \\ \tilde{A} \end{bmatrix}$, $\bar{u}^h$ is the utility of investor h in equilibrium in economy $\mathcal{E}$, and q is an equilibrium price vector in the economy $\mathcal{E}$. That is, $EV(A^{J+1})$, measures the total amount investors would want to pay for the new asset under *current* equilibrium prices.

4 A Numerical Analysis

In this section, a numerical analysis is presented to study the performance of the different valuation approaches. We consider a market with, initially, $J = 2$ assets and $S = 500$ states of nature.[6] The first asset is a riskless asset with constant payoffs across states,

$$A^1 = \mathbb{1} \in \mathbb{R}^S.$$

The other asset has a risky payoff stream, A^2. The latter asset can be thought of as representing the market portfolio. The riskless asset is in zero supply, whereas the risky asset is in unit supply.

Each investor is assumed to have a von Neumann–Morgenstern CRRA utility function, exhibiting habit formation (cf. Abel (1990))

$$u^h(x) = \frac{x_0^{1-\gamma}}{1-\gamma} + \frac{\delta}{1-\gamma}\mathbb{E}_P\Big[\Big(\frac{x_1}{x_0^\lambda}\Big)^{1-\gamma}\Big],$$

[6]With $S = 500$, we obtain a close match of the calibrated moments introduced below.

where $\delta \in (0,1)$ is the discount rate, $\gamma \geq 0$ is the coefficient of relative risk aversion, and $\lambda \geq 0$ is the coefficient of habit formation. Note that this gives a standard CRRA utility function if $\lambda = 0$. If $\lambda > 0$, preferences exhibit habit formation, in the sense that past consumption influences future utility.

Endowments at $t = 1$ consist of future labour income $(l_s^h)_{h=1,\ldots,H;s=1,\ldots,S}$ and dividend payoffs from initial portfolios, $(\bar{\theta}_j^h)_{h=1,\ldots,H;j=1,\ldots,J}$.[7] Total labour income is denoted by $L = \sum_{h=1}^{H} l^h$. It is assumed that the share of labour income of agent h, $\phi^h = \frac{l^h}{L}$, is constant over time and across states. Endowments at $t = 0$ consist solely of labour income and are taken to be unity for all h. Total labour income and the dividends on the risky asset are assumed to be bivariate log-normally distributed,

$$\begin{bmatrix} L \\ A^2 \end{bmatrix} \sim LN_2\Big(\begin{bmatrix} \mu_l \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma_l^2 & \rho_{l2} \\ \rho_{l2} & \sigma_2^2 \end{bmatrix} \Big).$$

This concludes the description of the economy $\mathcal{E}$.

In the simulations we have taken $\mu_l = 1.008$ and $\sigma_l = 0.008$, which corresponds to the mean and standard deviation of the (quarterly) US GDP-growth rate over the period 1978–2005. Taking the (quarterly) growth rate of the S&P500 over the same period gives $\mu_2 = 1.025$ and $\sigma_2 = 0.058$. The correlation between US GDP and the S&P500 over this period was $\rho_{l2} = 0.129$.

The new asset is assumed to be correlated with GDP. The correlation coefficient is denoted by ρ_{l3}. Via this channel, the new asset's payoffs are also correlated with the payoffs of the market portfolio. It is assumed that

$$\begin{bmatrix} L \\ A^3 \end{bmatrix} \sim LN_2\Big(\begin{bmatrix} \mu_l \\ \mu_3 \end{bmatrix}, \begin{bmatrix} \sigma_l^2 & \rho_{l3} \\ \rho_{l3} & \sigma_3^2 \end{bmatrix} \Big).$$

We study an economy with $H = 2$ households. To study the comparative statics of $GEI(A^3)$, we take as a baseline case $\mu_3 = 1.1$, $\sigma_3 = 0.1$, and $\rho_{l3} = 0$. The shares of labour income and initial portfolios are taken to be $\phi^1 = 1 - \theta^1 = \theta^2 = 1 - \phi^2 = 0.8$. It is assumed that preferences are identical, with $\gamma = 3$ and $\lambda = 0$. The comparative statics for several parameters are depicted in Figure 1. As becomes clear from Figure 1, increased risk aversion and habit formation have a positive effect on the asset's value, although the effect of λ is relatively small. The effects of the correlations between labour income and the market portfolio and the project, respectively, are negligible.

To study the performance of PV and CC valuation in this environment we have conducted 200 simulations of the model. In all simulations we have taken $\phi^1 = 1 - \theta^1 = \theta^2 = 1 - \phi^2 \equiv \theta$. The parameters have been sampled from the intervals as shown in Table 1. Over 200 simulation runs, the average R^2 of regressing A^3 on

[7] Note that introducing a new asset, therefore, also has an effect on initial endowments.

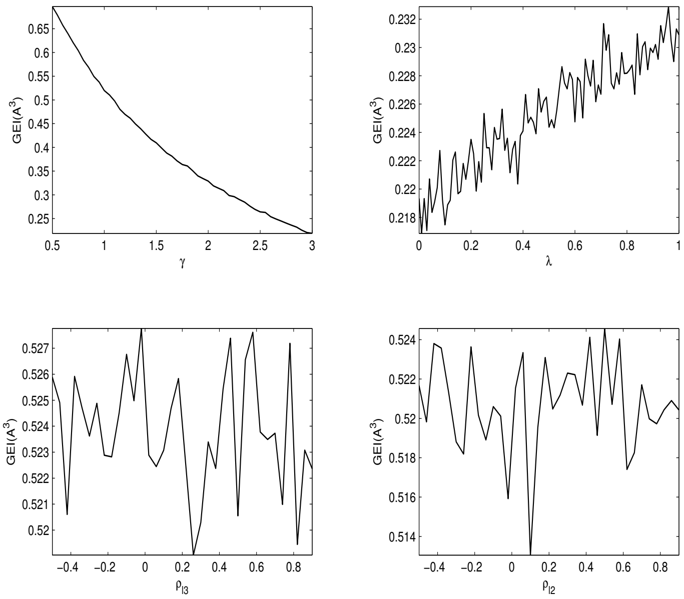

Figure 1: Comparative statics of $GEI(d^p)$ with respect to γ (top-left), λ (top-right), ρ_{l3} (bottom-left), and ρ_{l2} (bottom-right).

parameter	δ	γ	λ	μ_3	σ_3	ρ_{l3}	θ
interval	(0.9,1)	(0.5,3)	(0,1)	(1.03,1.4)	(0.01,0.2)	(-0.5,0.9)	(0.5,1)

Table 1: Simulation intervals for different parameters

$\langle A \rangle$, where $A = \begin{bmatrix} A^1 & A^2 \end{bmatrix}$, equals 0.008, with a standard deviation equal to 0.008. In other words, the new asset is virtually perpendicular to the market subspace.

The performance of CC (PV) is measured by regressing $GEI(A^3)$ on a constant and $CC(A^3)$ $(PV(A^3))$. In the case of PV this gives $R^2 = 0.022$, whereas for CC it results in $R^2 = 0.99$. In other words, PV is a very bad predictor of the equilibrium value of a project, whereas CC performs extremely well. To see how $GEI(A^3)$ is distributed over the various simulation runs, we have fitted a nonparametric Epanechnikov kernel density estimator using a bandwidth of 0.165, which minimises the Mean Integrated Square Error .The resulting density function is depicted in Figure 2. One can see that this distribution is considerably left-skewed and unimodal around its mean.

Interestingly enough, throughout the simulations, it holds that $EV(A^3) = 0$, and that utility for both agents is the same in $\mathcal{E}$ and $\tilde{\mathcal{E}}$. This means that the value of the new asset will immediately be incorporated in the equilibrium prices. The

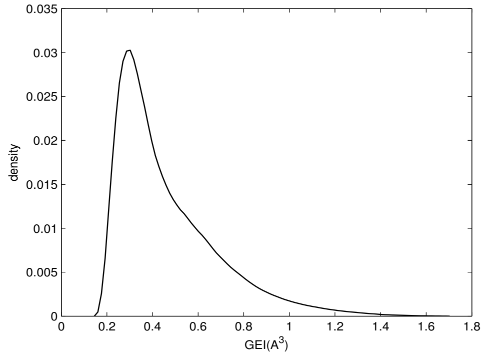

Figure 2: Epanechnikov kernel density estimate of $GEI(A^3)$.

value of a new asset is, therefore, solely measured by the extent to which it increases the possibilities of risk-sharing among agents and this value is priced in equilibrium. Risk-averse agents will attach a positive price to increased risk-sharing. Adding a new asset to the market will, therefore, not increase utility levels for risk-averse agents.

Another interesting feature of the simulations is the extent to which the new asset mitigates the incomplete markets problem. A useful measure of market completeness is the fraction

$$\alpha = \frac{var(proj_{\langle A \rangle \omega})}{var(\omega)},$$

i.e. the fraction of the variation in initial endowments over future states, which is traded on financial markets. That is, the fraction of variation in initial endowments that can be hedged. In $\mathcal{E}$ this fraction is on average (standard deviations between brackets) 0.984 (0.001), whereas in $\tilde{\mathcal{E}}$ it equals 0.987 (0.004). In other words, the additional asset does not improve market completeness very much. Even more surprising is the high level of market completeness in $\mathcal{E}$. This can be explained from the fact that assets are part of the initial endowments.[8]

5 Discussion

This paper studied financial innovation or investment appraisal in a two-period economy with heterogeneous risk averse agents and incomplete financial markets. In a computational analysis we compared standard measures of asset value, namely present value and contingent claim value (the latter is used in the currently popular

[8]A re-run of the simulations without initial asset holdings shows, indeed, a much lower degree of market completeness. In some cases, market completeness is significantly higher in $\tilde{\mathcal{E}}$. The very close fit of CC, however, does not change.

no-arbitrage pricing theory and real option theory) with actual equilibrium changes in both market and shareholder value. Our model has three important features:

1. investors are risk averse and heterogeneous in initial endowments, rendering obsolete standard risk-neutral, representative agent analysis;

2. financial markets are incomplete, resulting in, generically, non-replicable contingent claims;

3. financial market structure is endogenous, i.e. the asset payoff matrix is changed by the new asset, implying that contingent claim analysis might systematically find the wrong replicating portfolio.

In CC analysis one does take risk sharing opportunities into account. However, an error occurs since one projects the change in dividend streams on the market space prior to investment, whereas one should take into account the change in the market subspace due to the investment. The simulations show, however, that this error is negligible. This provides evidence that general equilibrium effects might be very small, so that standard (real) option techniques can safely be used.

Endogeneity of the market subspace implies that classical (frequentist) statistical methods should be used in investment appraisal with a high degree of caution, much in line with the Lucas critique. Standard procedures can be used to estimate the distribution of risk attitudes on the market, as those are unlikely to change much over time. The endogenous change to the market, however, is much less clear. Usually, one only has prior beliefs (in the Bayesian sense) over the influence of a structural change. This means that one should be careful in judging managers *ex post* on the basis of information (asset prices) that the manager did not have *ex ante*.

One can take this point even further. Since most empirical research in asset pricing uses cross-sectional analysis based on the iid assumption,[9] such studies might make a systematic error either if changes in the composition of listed firms occur, or if listed firms engage in investment projects during the sample period. In both cases, the market portfolio is likely to change and, hence, the underlying probability space changes.

Note that the two-period GEI model cannot be used to study financial option prices. Firstly, an option's payoff depends on the (equilibrium) price of an asset. Secondly, options are used to hedge risks in the face of uncertainty reducing over time. Both points suggest that at least three periods are needed to model financial options in a general equilibrium context. Several technical problems then arise,

[9] See, for example, Campbell et al. (1997) for an overview.

however, rendering such an analysis non-trivial.[10]

Finally, it is important to note that the analysis does not focus on another important aspect of financial innovation. The introduction of a new asset, namely, might change the span of the existing assets. For example, the IPO of a technology firm might influence the payoff streams of existing technology firms. An analysis of this issue, however, requires explicitly modelling the production side of a GEI economy. This, however, is outside the scope of the current paper and is left to future research.

Appendix

A Homotopy Methods in GEI Analysis

The equilibria in this paper are computed by using a homotopy method developed in Herings and Kubler (2002). This is a differentiable homotopy obtained by replacing excess demand functions by the first order conditions of utility maximisation, an approach proposed by Garcia and Zangwill (1981). The advantage of this approach is that the number of agents, H, and the number of assets, J, determine the dimensionality of the homotopy instead of the number of states, S, which is typically very large. Furthermore, there is no need to explicitly compute agents' demand function and the set of no-arbitrage prices Q, both typically non-trivial. Instead one merely needs the Jacobians of the utility functions.

The Herings-Kubler (HK) homotopy is designed for two-period GEI economies with no consumption at $t = 0$. A standard way of transforming any GEI economy $\mathcal{E}$ to an economy with no consumption at time is presented in Hens (1991) and consists of replacing the asset payoff matrix A by the matrix

$$\bar{A} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & A_1^1 & \dots & A_1^J \\ \vdots & \vdots & \ddots & \dots \\ 0 & A_S^1 & \dots & A_S^J \end{bmatrix}.$$

That is, $t = 0$ consumption is translated to $t = 1$. by introducing an artificial state $s = 0$ and an artificial asset $j = 0$. The analysis that follows is concerned with the economy $\bar{\mathcal{E}} = (u, \omega, \bar{A})$.

The following additional assumption is made with respect to investors' preferences.

[10]See, for example, Magill and Quinzii (1996, Chapter 4) for an exposition of these problems.

Assumption A.1 *For all $h = 1, \ldots, H$, the utility function u^h is three times continuously differentiable such that for all $x \in \mathbb{R}^S_{++}$ it holds that*

1. $\partial u^h(c) \in \mathbb{R}^S_{++}$;
2. $\forall_{y \neq 0: \partial u^h(c) y = 0} : y \top \partial^2 u^h(c) y < 0$;
3. $\{\tilde{c} \in \mathbb{R}^S_{++} | u^h(\tilde{c}) \geq u^h(c)\}$ *is closed in* $\mathbb{R}^S$.

Let $\hat{f}$ and $\hat{q}$ denote the excess demand function and an asset price system, respectively, with the first entry removed. The algorithm starts from an initial price system q^0 in Q, with the price of $t = 0$ consumption normalised to 1. Equilibria of $\bar{\mathcal{E}}$ are computed by the homotopy $\mathcal{H} : [0, 1] \times Q \to \mathbb{R}^J$

$$\mathcal{H}(t, q) = t\hat{f}(q) + (1 - t)(\hat{q}^0 - \hat{q}). \tag{A.1}$$

Herings and Kubler (2002) prove the following theorem.

Theorem A.1 *Let $\Omega \subset \mathbb{R}^{HS}_{++}$ be an open set with full Lebesgue measure. For all initial endowments $\omega \in \Omega$ it holds that*

1. $\mathcal{H}^{-1}(\{0\})$ *is a compact C^2 one-dimensional manifold with boundary* $\mathcal{H}^{-1}(\{0\}) \cap (\{0, 1\} \times Q)$;
2. *there is an odd number of solutions in* $\mathcal{H}^{-1}(\{0\}) \cap (\{1\} \times Q)$;
3. *there is one solution in* $\mathcal{H}^{-1}(\{0\}) \cap (\{0\} \times Q)$;
4. *there is no sequence $(t^n, q^n)_{n \in \mathbb{N}}$ in $\mathcal{H}^{-1}(\{0\})$ with limit $(t, q) \in [0, 1] \times \partial Q$ or such that $||(t^n, q^n)||_2 \to \infty$.*

That is, generically, there exists a path from q^0 to an FME q; there is only one solution at q^0; there is an odd number of solutions; and the algorithm does not diverge or converge to the boundary.

The homotopy $\mathcal{H}$ has the advantage that one does not have to compute the set Q explicitly. Unfortunately, however, it is usually non-trivial to compute the excess demand function f analytically. One can use (A.1), but at every step n, the function value $\hat{f}(q^n)$ has to be computed numerically, which is highly time consuming. Instead one can replace $\mathcal{H}$ with the diffeomorphic implementable homotopy $\mathcal{H}^* : [0, 1] \times Q \times \mathbb{R}^{H(J+1) \times \mathbb{R}^H} \to \mathbb{R}^{(H+1)(J+2)-2}$, defined by

$$\mathcal{H}^*(t, q, \theta, \lambda) = \begin{cases} t \sum_{h=1}^H \theta^h_j + (1 - t)(q^0_j - q_j), & j = 1, \ldots, J \\ \left(\partial u^h(\omega^h + \bar{A}\theta^h)\bar{A}\right)^\top - \lambda^h q^\top, & h = 1, \ldots, H \\ q\theta^h, & h = 1, \ldots, H, \end{cases} \tag{A.2}$$

where λ is the vector of Lagrange multipliers obtained from utility maximisation. We have the following result.

Theorem A.2 (Herings and Kubler (2002)) *$(\mathcal{H}^*)^{-1}(\{0\})$ is C^2 diffeomorphic to $\mathcal{H}^{-1}(\{0\})$.*

This implies that, generically, the homotopy $\mathcal{H}^*$ converges to an FME. The homotopy (A.2) is implemented in Matlab via a four-step Adams-Bashforth predictor-corrector method.

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