A Stochastic Discount Factor Approach to Investment under Uncertainty*

Jacco Thijssen†

January 2006

Abstract

This paper presents a unified approach to valuing investment projects under uncertainty. It is argued that the most important aspect of investment appraisal is the choice of discount factor. An investment threshold for the case where the discount factor and the project's cash-flows both follow a geometric Brownian motion is derived. Numerical results on the comparative statics of the threshold are obtained. The paper illustrates how discount factors can be obtained both from a preference-based and from a markets-based (no-arbitrage) approach. The paper extends the latter approach to valuing projects in incomplete markets, by using good-deal bounds.

*Helpful comments from Peter Kort and seminar participants at University College Dublin are much appreciated. Alastair Ross is thanked for research assistance in the early stages of this project. Funding from the Irish Research Council for the Humanities and Social Sciences (IRCHSS) is gratefully acknowledged. The usual disclaimer applies.

†Department of Economics, University of Dublin, Trinity College, Dublin 2, Ireland. Phone: +353(0)16081569, email: Jacco.Thijssen@tcd.ie.

1 Introduction

Recent years have seen a wide variety of theoretical and applied contributions in the area of investment under uncertainty. The current paradigm views an investment project as a (real) option. Following the seminal contribution of McDonald and Siegel (1986), many applications have been studied in the literature. See, for example, Dixit and Pindyck (1994) and Trigeorgis (1996) for an overview of the early literature. Real option theory (ROT) has been developed in reaction to some shortcomings in the, still popular, standard theory of investment appraisal: the net present value (NPV) approach. This approach prescribes that investment should take place if, and only if, the expected value of the discounted future cash flows exceeds the sunk investment costs. Two problems that arise from applying NPV are, firstly, that many investment projects are (at least partially) irreversible. Secondly, most investment decisions can be postponed until more information has been obtained. This creates an option value of waiting, which NPV does not take into account. By modelling an investment project like an American call option the standard ROT model concludes that an investor should wait longer than prescribed by NPV for investment to be optimal.

In the latter half of the nineties, attention spread to a game theoretic analysis of investment timing under uncertainty. Following the seminal paper by Smets (1991), many papers followed modelling investment timing games as preemption games (following Fudenberg and Tirole (1985)), wars of attrition (following Hendricks et al. (1988)), or combinations of both. Notable contributions are Grenadier (2000), Huisman (2001), Weeds (2002), Thijssen et al. (2006). See Huisman et al. (2004) for an overview.

Most of the contributions within the framework of ROT assume risk-neutral investors and complete markets. In recent years, the problem of investment in incomplete markets has received some attention. These papers usually assume a risk-averse investor and model the investment decision using CARA or CRRA utility functions. Recent papers are, for example, Van den Goorbergh et al. (2003), Hugonnier and Morellec (2004, 2005), and Miao and Wang (2005).

This paper attempts to provide a unified approach, which encompasses most of the (single decision-maker) real options literature. The main aim is to clarify the underlying principles, which appear under the surface in most real options models. The tool used here is the concept of a *(stochastic) discount factor*. Recently, it has been argued in the asset pricing literature that many models in financial economics, like CAPM, APT, and arbitrage pricing, are merely special cases of a general model where assets are priced relative to a discount factor (cf. Cochrane (2005)).

For real option theory, a similar project can be undertaken. Given that a firm has decided to value an investment project relative to a discount factor, an investment threshold can be derived. In deriving this threshold, at each point in time the firm has to compare the net present value of investing immediately and the option value of postponing investment. Both the net present value and the option value are computed relative to the discount factor. Moreover, the option value is assumed to follow Bellman's principle of optimality, which leads to a no-arbitrage value for the option to invest.

The basic premise in corporate finance is that a firm should take investment decisions that are "in the interest of the shareholders". The value of an investment project depends crucially on the chosen discount factor. Therefore, the discount factor should represent the time and risk preferences of the shareholders. In this paper, a distinction is made between two different methods of obtaining a discount factor and, hence, of modelling the interests of the shareholders. These two approaches both appear in the standard ROT literature, although usually no explicit discount factor is derived. Firstly, one can follow a *preference-based* approach and assume that there exists a representative investor who has preferences satisfying the axioms of expected utility theory. Alternatively, a *market-based* approach can be chosen. Here one uses the price processes of traded assets to obtain a discount factor, based on the assumption that asset prices are arbitrage-free. Note that in the latter approach it is irrelevant what the actual preferences of the investors are. Furthermore, it is not important whether investors are rational utility maximising agents. The two approaches could, therefore, alternatively be coined *subjective* and *objective*, respectively.

The market-based approach is only well-defined in complete markets. In incomplete markets the traded assets do not provide a unique discount factor that spans all the risk in the project's cash-flows. As mentioned above, most recent contributions then resort to representative agent analysis. That is, one replaces the objective approach with the subjective approach, in which market incompleteness plays no role. However, even though no unique discount factor can be found, it is, nevertheless, possible to come up with reasonable bounds on the value of an investment project. In this paper, the concept of "good-deal bounds" (cf. Cochrane and Saá-Requejo (2000)) is used to derive an upper and lower investment threshold. The model values a project relative to a discount factor obtained from traded assets. However, since it is assumed that the cash flow uncertainty of the project is not fully spanned by the market, this discount factor is not unique. We obtain an upper and a lower bound for the value of the project under an exogenously given constraint, which bounds the instantaneous volatility of the discount factor. As in the complete markets case, Sharpe ratios play an important role. In this case, however, one also has to determine the loading of the discount factor on the idiosyncratic risk. It is shown that the correlation of the idiosyncratic shock with the risk of the traded assets is an important determinant of this loading.

The paper is organised as follows. In Section 2, the discount factor approach is developed. The resulting investment threshold is analysed in Section 3. In Section 4, two ways of obtaining a discount factor are discussed, namely the preference-based (or subjective) and the markets-based (or objective) approach. For the latter it is shown how good-deal bounds can be used to conduct investment appraisal in incomplete markets in Section 5. Finally, Section 6 discusses the results of the paper.

2 The Discount Factor and Investment Appraisal

This section introduces the general model. Time is assumed to be continuous and indexed by $t \geq 0$. Uncertainty is modelled by a filtered probability space $\mathcal{P} = \big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq \infty}, P\big)$. The firm can invest in a project, which leads to an uncertain stream of cash flows, $(V_t)_{0 \leq t < \infty}$, which is adapted to $\mathcal{P}$ and governed by the diffusion

$$\frac{dV_t}{V_t} = \mu_V dt + \sigma_V dz, \tag{1}$$

where $\mu_V \geq 0$ is the trend, $\sigma_V \geq 0$ is the instantaneous standard deviation, and $dz \sim \mathcal{N}(0, dt)$ is the increment of a Wiener process. The sunk costs are assumed to be constant and equal to $I > 0$.

As standard corporate finance theory prescribes (cf. Brealey and Myers (2003)), management should take investment decisions "in the interest of the shareholders". This is usually interpreted to imply that managers have to maximise the net present value (NPV). Real option theory (ROT) has convincingly argued that in addition one needs to include the option value of waiting for more information. An important question in capital budgeting is how to obtain the appropriate discount rate at which the stream of future cash flows should be discounted. If this stream can be replicated by a dynamic portfolio of traded assets, one can use standard contingent claims analysis to value investment projects. The appropriate discount rate is then obtained by applying, for example, the Capital Asset Pricing Model (CAPM), i.e. by linking the project's cash flow stream to returns on the "market portfolio". This implies that one assumes that there is a representative investor with mean-variance preferences. Alternatively, an exogenously given discount rate could be used. Both approaches, however, can be seen as special cases of a more general method where investment projects are valued relative to a (stochastic) discount factor.

Definition 1 (Discount Factor) *A process* $(\Lambda_t)_{0\leq t<\infty}$ *is a discount factor if it is adapted to* $\mathcal{P}$ *and if* $\Lambda_t > 0$*, for all* $t \geq 0$*.* $\triangle$

The management of a firm should value each investment project relative to a discount factor, which it beliefs represents the interest of the shareholders. Investment should then take place at an optimal time, in the sense defined below. In the current formulation, V is a sufficient statistic for the project's net present value, which is increasing in V. Therefore, the optimal investment policy can be represented by a stopping time. Denote by $\mathcal{T}$, the set of stopping times T, adapted to $\mathcal{P}$, that are of the form $T(\omega) = \inf\{t \geq 0 | V_t \geq V^*\}$, for some threshold V^*. Let[1]

$$F(V_t) = \frac{1}{\Lambda_t}\mathbb{E}_t\Big(\int_t^\infty \Lambda_s V_s ds - \Lambda_t I\Big),$$

be the net present value of the project if investment takes place at time $t \geq 0$.

Definition 2 (Optimal investment time) *A stopping time* $T^* \in \mathcal{T}$ *is optimal with respect to a discount factor* $(\Lambda_t)_{0\leq t<\infty}$ *if it solves the problem*

$$F(V_{T^*}) = \sup_{T\in\mathcal{T}} \mathbb{E}_T F(V_T). \tag{2}$$

$\triangle$

Before investment takes place, the value of the project consists of the value of the *possibility* to invest at a certain time in future. This is, in other words, the *option value*, which is denoted by $C = C(V)$. At each time $t \geq 0$, the value of the project then equals

$$\Phi(V_t) = \max\{F(V_t), C(V_t)\}.$$

The option value can, in principle, be derived in many ways. It is standard to apply *Bellman's principle of optimality.*

Definition 3 (Bellman principle) *Let* $(\Lambda_t)_{0\leq t<\infty}$ *be a discount factor and let* $\tau > 0$ *be the time of investment. The value process* C *satisfies the Bellman principle on the interval* $[0,\tau)$ *with respect to* Λ *if, for all* $0 \leq t < \tau$*, it holds that,*

$$\Lambda_t C_t = \lim_{dt\downarrow 0} \mathbb{E}_t(\Lambda_{t+dt} C_{t+dt}).$$

$\triangle$

In differential form the Bellman principle can be written as

$$\mathbb{E}_t(d\Lambda_t C_t) = 0.$$

[1]In the remainder, for any random variable X, we denote $\mathbb{E}_t(X) := \mathbb{E}(X|\mathcal{F}_t)$.

In financial economics the Bellman principle is usually referred to as the *no-arbitrage principle*, which stipulates that the value of a (contingent) claim should be such that it does not open up arbitrage opportunities. In ROT, this principle can be illustrated as follows. Suppose that the project is a traded asset. Consider a European call option written on this asset with time to maturity τ and strike price $F(V_\tau)$. Suppose, furthermore, that an investor has a dynamic portfolio of this option, $(\Lambda_t)_{0\leq t<\infty}$. At time $t < \tau$, the price of the option, $C_t = C(V_t)$, is said to satisfy the no-arbitrage principle if the dynamic portfolio Λ does not yield a profit in expectation, i.e. if the value of the portfolio at time t is equal to the expected value of the portfolio at time $t + dt$:

$$\Lambda_t C_t = \lim_{dt\downarrow 0} \mathbb{E}_t(\Lambda_{t+dt}C_{t+dt}).$$

In the remainder of this section it is assumed that a discount factor $(\Lambda_t)_{0\leq t<\infty}$ is given, which follows the diffusion

$$\frac{d\Lambda}{\Lambda} = -\mu_\Lambda dt - \sigma_\Lambda dz, \tag{3}$$

where $\mu_\Lambda > 0$ and $\sigma_\Lambda \geq 0$. Note that this formulation implies that dV and $d\Lambda$ are perfectly correlated. In Section 4 two ways of obtaining a discount factor are described.

If investment takes place at time τ, the net present value of the project relative to Λ, equals

$$F(V_\tau) = \mathbb{E}_\tau\Big(\int_\tau^\infty \frac{\Lambda_t}{\Lambda_\tau} V_t dt\Big) - I. \tag{4}$$

From Ito's lemma, it follows that

$$\begin{aligned}\frac{d\Lambda V}{\Lambda V} &= \frac{dV}{V} + \frac{d\Lambda}{\Lambda} + \frac{d\Lambda}{\Lambda}\frac{dV}{V}\\ &= -(\mu_\Lambda + \sigma_\Lambda\sigma_V - \mu_V)dt + (\sigma_V - \sigma_\Lambda)dz.\end{aligned}$$

Therefore (cf. Huisman (2001, Chapter 7)),

$$F(V_\tau) = \frac{V_\tau}{\delta} - I,$$

where

$$\delta = \mu_\Lambda + \sigma_\Lambda\sigma_V - \mu_V,$$

is the so-called *convenience yield*. If $(V_t)_{0\leq t<\infty}$ were the price process of a stock, then δ would represent its dividend rate. The option to invest would then be an American call option on this stock with an infinite time to maturity. If $\delta > 0$, the option will not be held indefinitely, but exercised at the optimal stopping time T^*. The parameter δ reflects (Dixit and Pindyck (1994, p. 149)) "an opportunity cost of delaying construction of the project, and instead keeping the option to invest alive". Note that $F(V_\tau) < \infty \iff \delta > 0 \iff \mu_V < \mu_\Lambda + \sigma_\Lambda \sigma_V$. In the remainder it is assumed that this condition holds.[2]

Suppose that investment takes place at time $\tau > 0$. It is assumed that the value of the option to invest, C, is a twice continuously differentiable function. This value, too, has to be valued with respect to the discount factor Λ. Imposing the Bellman principle gives that C should be such that

$$\mathbb{E}_t(d\Lambda C) = 0 \iff \mathbb{E}_t(dC) + \mathbb{E}_t\Big(\frac{d\Lambda}{\Lambda}\Big) = -\mathbb{E}_t\Big(\frac{d\Lambda}{\Lambda}dC\Big), \tag{5}$$

for all $0 \leq t < \tau$.

From Ito's lemma it follows that[3]

$$\begin{aligned} dC &= C'_V dV + \frac{1}{2}C''_{VV}(dV)^2 \\ &= (\frac{1}{2}\sigma_V^2 V^2 C''_{VV} + \mu_V V C'_V)dt + \sigma_V V C'_V dz. \end{aligned} \tag{6}$$

Furthermore,

$$\frac{d\Lambda}{\Lambda}dC = \sigma_\Lambda \sigma_V V C'_V dt. \tag{7}$$

Substituting (6) and (7) into (5) and rearranging yields the partial differential equation (PDE)

$$\frac{1}{2}\sigma_V^2 V^2 C''_{VV} + (\mu_\Lambda - \delta)V C'_V - \mu_\Lambda C = 0. \tag{8}$$

The solution to this PDE is given by

$$C(V) = \eta_1 V^{\beta_1} + \eta_2 V^{\beta_2},$$

where η_1 and η_2 are constants, and β_1 and β_2 are the roots of the quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_V^2 \beta(\beta - 1) + (\mu_\Lambda - \delta)\beta - \mu_\Lambda = 0. \tag{9}$$

Note that $\mathcal{Q}(\cdot)$, and hence the solutions to $\mathcal{Q}(\beta) = 0$, depend on all parameters of the model. Given the assumptions on μ_Λ and finiteness of $F(V_t)$, it holds that $\mathcal{Q}(0) < 0$ and $\mathcal{Q}(1) <$ and, hence, that $\beta_1 > 1$ and $\beta_2 < 0$.

The optimal stopping problem (2) can now be solved after imposing the appropriate boundary conditions. These are:

[2] The intuitive idea behind this condition is that if $\delta < 0$, then the cash-flows of the project are have a higher expectation and a lower risk than $-\Lambda$. Therefore, if the firm had a choice it would want to invest in the discount factor and never in the project.

[3] In the remainder, for any function $f : \mathbb{R}^n \to \mathbb{R}$, we denote $f'_{x_i}(\cdot) := \frac{\partial f(\cdot)}{\partial x_i}$ and $f''_{x_i x_j}(\cdot) := \frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j}$.

1. No speculative bubbles: $\lim_{V \to 0} C(V) = 0$;

2. Value-matching: $C(V^*) = F(V^*)$;

3. Smooth-pasting: $C'_V(V^*) = F'_V(V^*)$.

Since $\beta_2 < 0$, the first condition implies that $\eta_2 = 0$. The investment threshold, V^*, and constant, η_1, are found by simultaneously solving the value-matching and smooth-pasting conditions, yielding

$$V^* = \frac{\beta_1}{\beta_1 - 1}\delta I \quad \text{and} \quad \eta_1 = \frac{1}{\beta_1}\frac{(V^*)^{\beta_1}}{\delta}. \tag{10}$$

Investment takes place at time $T^* = \inf\{t \geq 0 | V_t \geq V^*\}$. The value of the project, $\Phi(\cdot)$, is, therefore, equal to

$$\Phi(V) = \begin{cases} \frac{1}{\beta_1 \delta}(V^*)^{1-\beta_1} V^{\beta_1} & \text{if } V < V^*; \\ \frac{V}{\delta} - I & \text{if } V \geq V^*. \end{cases}$$

Note that $\Phi(\cdot)$ is continuous and differentiable. To summarise, we have proved the following proposition.

Proposition 1 *Let $(V_t)_{0 \leq t < \infty}$ be an Ito diffusion following (1) and let $(\Lambda_t)_{0 \leq t < \infty}$ be a discount factor following (3). If it holds that $\mu_\Lambda > 0$, and $\mu_V < \mu_\Lambda + \sigma_\Lambda \sigma_V$, then the optimal stopping problem (2) is solved by $T^* = \inf\{t \geq 0 | V_t \geq V^*\}$, where*

$$V^* = \frac{\beta_1}{\beta_1 - 1}\delta I,$$
$$\delta = \mu_\Lambda + \sigma_\Lambda \sigma_V - \mu_V,$$

and β_1 is the positive root of

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_V^2 \beta(\beta - 1) + (\mu_\Lambda - \delta)\beta - \mu_\Lambda = 0.$$

The value of the investment project equals $\Phi(\cdot)$. ▲

3 Analysis of the Investment Threshold

In this section we analyse the influence of the different parameters on the investment threshold V^*. Let $\zeta \in \{\mu_\Lambda, \sigma_\Lambda, \mu_V, \sigma_V\}$ be one of the parameters of the model. The partial derivative of V^* with respect to ζ is then equal to

$$\begin{aligned} \frac{\partial V^*}{\partial \zeta} &= \Big(\frac{\beta_1}{\beta_1 - 1}\frac{\partial \delta}{\partial \zeta} + \frac{\partial}{\partial \beta_1}\frac{\beta_1}{\beta_1 - 1}\frac{\partial \beta_1}{\partial \zeta}\delta\Big) I \\ &= \Big(\frac{1}{\delta}\frac{\partial \delta}{\partial \zeta} - \frac{1}{\beta_1(\beta_1 - 1)}\frac{\partial \beta_1}{\partial \zeta}\Big) V^*. \end{aligned} \tag{11}$$

This expression shows that two effects influence the comparative statics of V^*. The first term between brackets in (11) is the so-called *present-value effect*. It measures how the threshold changes due to a change in the discount rate, δ. Note that the present value effect is insensitive to risk (σ_V) if the discount factor is deterministic ($\sigma_\Lambda = 0$).[4] The second term refers to the *option value effect*. This measures the change in the option value (or the value of waiting) of the project before investment takes place. Together, these two effects determine the total effect.

In most cases the two effects are of an opposed sign, as Table 1 shows, where (+) denotes a positive effect and (−) denotes a negative effect. Note that the positive

ρ	Present-value effect	Option value effect
μ_V	$-$	$+$
σ_V	$+$	$\begin{cases} + & \text{if } \sigma_V > \frac{\sigma_\Lambda}{\beta_1 - 1} \\ - & \text{if } \sigma_V < \frac{\sigma_\Lambda}{\beta_1 - 1} \end{cases}$
μ_Λ	$+$	$-$
σ_Λ	$+$	$-$

Table 1: Present value and option value effects.

present value effect reported for σ_V is only strictly positive for $\sigma_\Lambda > 0$. In case of a deterministic discount factor, a change in σ_V has no influence on the present value.

In order to see which of the two effects dominates, consider Figure 1. As a baseline case, the parameter values have been chosen as in Table 2. The sunk investment cost is taken to be $I = 100$. In all cases the present value effect outweighs the option value effect.

The only ambiguous effect is in the option value effect for σ_V. Figure 2 depicts the comparative statics of V^* and β_1 for σ_V. The latter is indicative of the option value effect. As one can see the option value effect is positive first and then turns negative. This, however, has no influence on the positive overall effect on V^* of increasing risk. The question that arises is whether there are parameter specifications where the investment threshold is decreasing in σ_V. Obviously, this can only happen when

[4] In fact, this is the main difference between the standard risk-neutral approach and the more recent literature on risk aversion. See Section 4.

$\mu_V = 0.05$	$\mu_\Lambda = 0.05$
$\sigma_V = 0.1$	$\sigma_\Lambda = 0.1$

Table 2: Parameter values

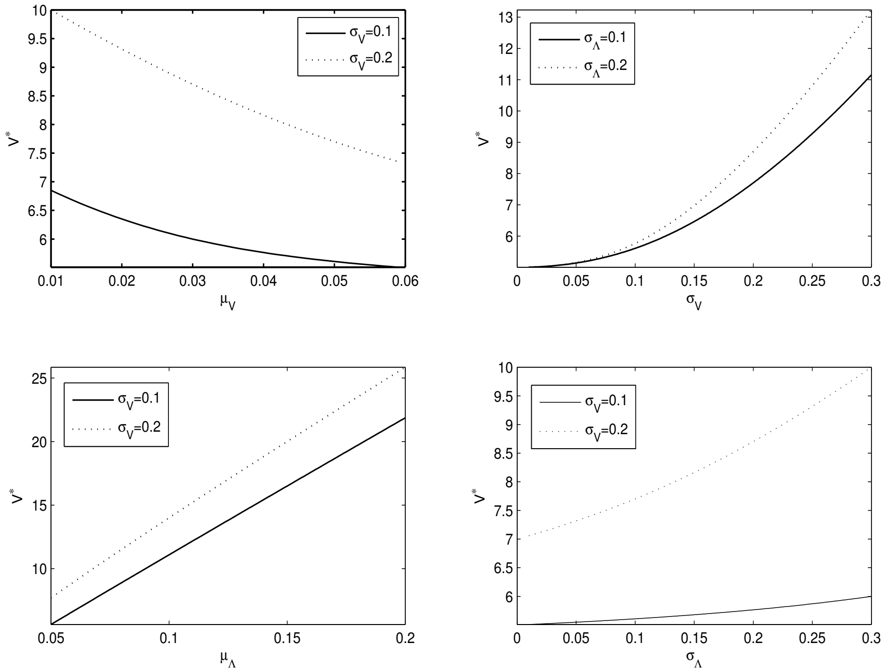

Figure 1: Comparative statics for the parameters μ_V, σ_V, μ_Λ, σ_Λ.

$\sigma_V < \frac{\sigma_\Lambda}{\beta_1 - 1}$. With $I = 100$, a simulation study has been conducted to investigate this issue. A total of 10^7 scenarios have been checked where μ_Λ, σ_V, and σ_Λ have been drawn uniformly from the intervals $(0, 0.2)$, $(0, 2)$, and $(0, 2)$, respectively. In each run, μ_V has been drawn uniformly from the interval $(0, \mu_\Lambda + \sigma_\Lambda \sigma_V)$. In none of the runs the threshold, V^*, was decreasing with respect to a small increase in σ_V ($\Delta\sigma_V = 10^{-4}$).

Finally, we study the influence of risk (σ_V) on the probability of investment taking place within, say, T periods. Let $\bar{\mu} = \mu_V - \frac{\sigma_V^2}{2}$. The probability that the threshold V^* is reached within T periods then equals

$$
\begin{aligned}
\mathbb{P}\left(\sup_{0 \le t \le T} V_t \ge V^*\right) =& \mathcal{N}\left(\frac{-\log(V^*/V_0) + \bar{\mu}T}{\sigma_V \sqrt{T}}\right) \\
&+ \left(\frac{V^*}{V_0}\right)^{\frac{2\bar{\mu}}{\sigma_V^2}} \mathcal{N}\left(\frac{-\log(V^*/V_0) - \bar{\mu}T}{\sigma_V \sqrt{T}}\right),
\end{aligned}
\tag{12}
$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution function of the standard normal distribution and V_0 is the initial value of the process $(V_t)_{0 \le t < \infty}$. Taking $T = 5$, $I = 100$, $\mu_V = \mu_\Lambda = 0.05$, and $\sigma_\Lambda = 0.3$, the comparative statics are depicted in Figure 3. As one can see, the probability is very sensitive to the initial value V_0. In fact, for a low initial value ($V_0 = 3$) the probability of investment is first increasing with risk

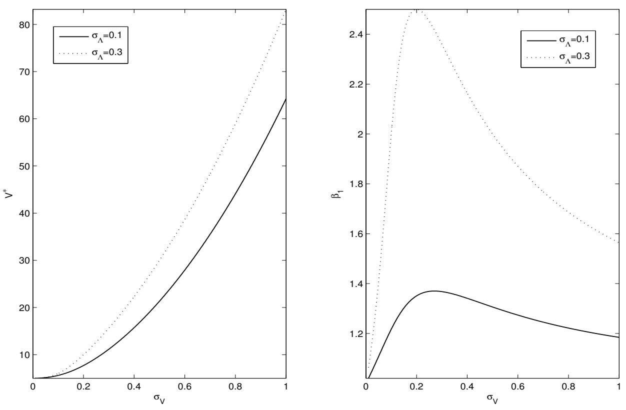

Figure 2: Comparative statics for σ_V (left panel) and β_1 (right panel).

for low values of σ_V, then decreasing for higher values of σ_V, and then increasing again for high values of σ_V. This behaviour can occur since there are two opposing effects. On the one hand the threshold increases with σ_V, which gives a downward pressure on the probability of investment. On the other hand, the threshold might be reached earlier, due to the increased volatility of V. From the numerical example it is clear that either effect may dominate.

4 Obtaining a Discount Factor

Throughout Section 3 it was assumed that a discount factor was exogenously given. In this section two different ways of obtaining a discount factor are discussed. The *preference-based* approach uses an exogenously specified utility function, where the discount factor is obtained from the first-order condition of utility maximisation. The second, or *market-based*, approach uses observed prices from traded assets to obtain a discount factor. As will be shown, Sharpe ratios play an important role in this method. Alternatively, one could classify the two approaches as the *subjective* and *objective* approach, respectively. Most models in the literature use a discount factor from one of these two categories. Essentially, though, both often lead to special cases of the model presented in Section 2, as is illustrated below.

4.1 The Preference-Based, or Subjective, Approach

One way to obtain a discount factor is to specify a utility function and a discount rate for the manager. These could be the preference characteristics of a representative shareholder (following Lucas (1978)). The manager then determines the time of

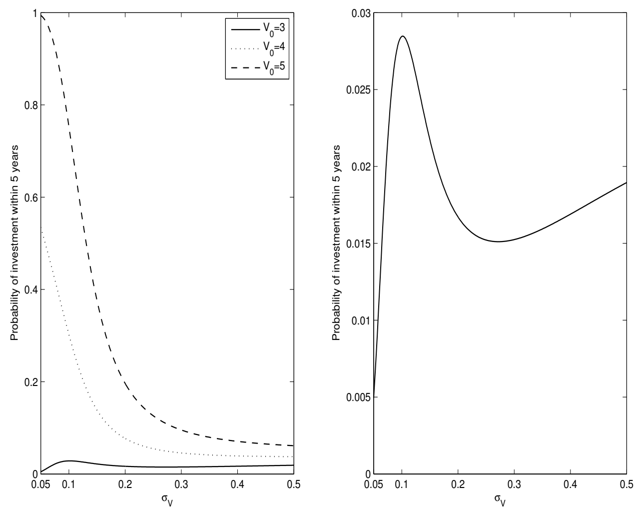

Figure 3: Probability of investment within 5 years.

investment such that it maximises the shareholder's expected discounted utility. This approach relies heavily on Expected Utility Theory and has, therefore, a well-axiomatised microeconomic foundation.[5]

Suppose there exists a representative investor with discount rate $\rho \in (0,1)$, and instantaneous utility function $u(V_t)$. Combining the discount rate and the utility function one can derive a discount factor at which the shareholder discounts the stream of cash-flows V. Consider a point in time $t \geq 0$, and a small time interval $(t, t+dt)$. Let Λ_t denote the discount factor at time t. That is, any change in the cash-flows, dV_t, has value $\Lambda_t dV_t$ to the shareholder. To be precise,

$$\begin{aligned}\Lambda_t dV_t &= \lim_{dt\downarrow 0}\left(e^{-\rho(t+dt)}u(V_{t+dt}) - e^{-\rho t}u(V_t)\right)\\ &= \lim_{dt\downarrow 0} e^{-\rho t}\big((1-\rho dt)u(V_{t+dt}) - u(V_t)\big)\\ &= \lim_{dt\downarrow 0}\left(e^{-\rho t}du - \rho u(V_{t+dt})dt\right)\\ &= e^{-\rho t}u'(V_t)dV_t.\end{aligned}$$

Hence, the firm acts in the interest of the representative shareholder if it solves the optimal stopping problem (2) with the discount factor $\Lambda_t = e^{-\rho dt}u'(V_t)$. For $t < T^*$, the Bellman principle then requires that the option value, $C = C(V)$, solves

$$\mathbb{E}_t(d\Lambda C) = 0.$$

Assume, for simplicity, that $u(V_t) = \frac{V_t^{1-\gamma}-1}{1-\gamma}$. That is, $u(\cdot)$ exhibits constant

[5]See, for example Mas–Colell, Whinston, and Green (1995) for a standard textbook exposition.

relative risk aversion (CRRA). Denoting $g(t, V_t) = e^{-\rho t} u'(V_t)$, applying Ito's lemma gives

$$\begin{aligned} d\Lambda &= \frac{\partial g(\cdot)}{\partial t} dt + \frac{\partial g(\cdot)}{\partial V} dV + \frac{1}{2} \frac{\partial^2 g(\cdot)}{\partial V^2} (dV)^2 \\ &= -\left(\rho + \gamma \mu_V - \frac{1}{2} \gamma(\gamma+1) \sigma_V^2\right) \Lambda dt - \gamma \sigma_V \Lambda dz. \end{aligned}$$

That is, $(\Lambda_t)_{0 \le t < \infty}$ follows a diffusion as in (3). The convenience yield equals $\delta = \rho - (1-\gamma)\mu_V + \frac{1}{2}\gamma(1-\gamma)\sigma_V^2$. Assuming that $\delta > 0$, Proposition 1 gives the investment threshold:

$$V^* = \frac{\beta_1}{\beta_1 - 1} \delta I,$$

and β_1 is the positive root of

$$\mathcal{Q}(\beta) \equiv \frac{1}{2} \sigma_V^2 \beta(\beta - 1) + (\mu_\Lambda - \delta)\beta - \mu_\Lambda = 0.$$

This solution is well-defined if $\rho > (1-\gamma)\mu_V - \frac{1}{2}\gamma\sigma_V^2(3+\gamma)$. That is, the representative shareholder has to be impatient enough. The standard risk-neutral model is a special case of this specification, namely if $\gamma = 0$. Then the threshold equals $V^* = \frac{\beta_1}{\beta_1 - 1}(\rho - \mu_V)I$, and is well-defined if $\rho > \mu_V$. Note that in this case it holds that $\sigma_\Lambda = 0$ and that, therefore, $\frac{\partial \delta}{\partial \sigma_V} = 0$, i.e. the net present value of the project is insensitive to the risk in the project's cash-flows.

Note that this simple model also includes more elaborate models that use multiplicative uncertainty. Let $D_1 > D_0 > 0$ be constants. Assume that, before and after investment, the firm's cash flows are $D_0 V$ and $D_1 V$, respectively. The investment project can then de interpreted as the firm investing I to change the stream of cash flows from $D_0 V$ to $D_1 V$. In terms of investment timing this is equivalent to an investment project which pays no dividends currently, but yields a stream of cash flows $(D_1 - D_0)V$ after investment. Note that $(D_1 - D_0)V$ follows a geometric Brownian motion with trend μ_V and standard deviation σ_V. Alternatively, one can easily see that this is equivalent to a project which yields cash flows V at sunk costs $\frac{I}{D_1 - D_0}$. Therefore, the optimal investment threshold is

$$V^* = \frac{\beta_1}{\beta_1 - 1} \frac{\delta}{D_1 - D_0} I,$$

which is equivalent to the standard threshold reported in the literature (cf. Huisman (2001, Chapter 7)).

A disadvantage of this approach is that the threshold is not necessarily monotonic in the degree of risk aversion. As an example, we take $I = 100$, $\rho = 0.1$, $\mu_V = 0.06$, and $\sigma_V = 0.2$. The left-panel of Figure 4 shows the threshold V^* for $\gamma \in [0, 3]$.

The right-panel of Figure 4 shows the probability of investment within 5 years as a function of γ. This probability, too, is non-monotonic in the degree of risk aversion.

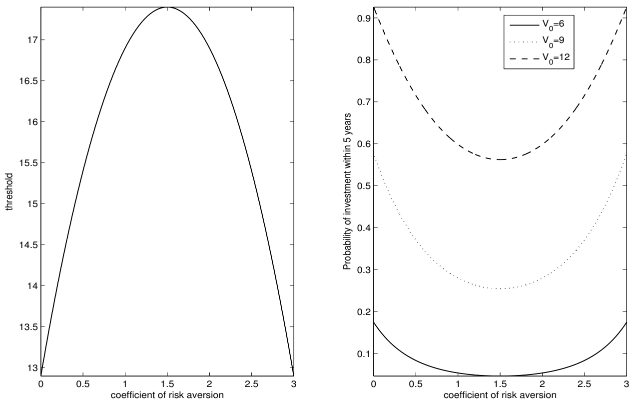

Figure 4: Comparative statics for γ of the investment threshold (left panel) and the probability of investment within 5 years (right panel).

4.2 The Market-Based, or Objective, Approach

Instead of embedding the discount factor in microeconomic theory, one can use the price processes of a set of assets to obtain a discount factor. One then assumes that the preferences of all investors are summarised in the prices. Note that it is irrelevant in this case whether investors are rational utility maximisers. One takes prices as given and is not worried about how these prices came about.

In this subsection it is assumed that the cash flow uncertainty is completely spanned by the asset price process. The firm can value the project relative to a riskless bond with price process $(B_t)_{0\leq t<\infty}$, and a risky asset with price process $(S_t)_{0\leq t<\infty}$.[6] Assume that B and S follow the Ito diffusions

$$\frac{dB}{B} = rdt \quad \text{and} \quad \frac{dS}{S} = \mu_S dt + \sigma_S dz, \tag{13}$$

Note that in this formulation it holds that $corr(dS/S, dV/V) = 1$, so that the uncertainty in V is spanned completely by S. It is easy to verify that there is a unique discount factor $(\Lambda_t)_{0\leq t<\infty}$, which values both B and S, satisfying the NA principle, which follows the diffusion

$$\frac{d\Lambda}{\Lambda} = -rdt - \frac{\mu_S - r}{\sigma_S} dz,$$

[6] In principle, this analysis can easily be extended to multiple risky assets. For expositional clarity, however, attention is restricted here to one risky asset.

where $h_S \equiv \frac{\mu_S - r}{\sigma_S}$ is the Sharpe ratio of asset S.

The convenience yield in this case equals $\delta = \sigma_V(h_S - h_V)$, where $h_V \equiv \frac{\mu_V - r}{\sigma_V}$ is the Sharpe ratio of the project. If one assumes that $h_V < h_S$, then Proposition 1 provides the optimal investment threshold:

$$V^* = \frac{\beta_1}{\beta_1 - 1}\sigma_V(h_S - h_V)I,$$

and β_1 is the positive root of

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_V^2\beta(\beta - 1) + (r - \delta)\beta - r = 0.$$

Two important issues arise from using the market-based approach. Firstly, it is impossible to value a project which has a Sharpe ratio larger than the Sharpe ratio of the risky asset relative to which one wants to value the project. This implies that very attractive projects (in the sense of having a high Sharpe ratio) can not be valued. Secondly, it matters which asset (or market index) is used. The same project will be valued differently relative to different assets. This can influence the probability of investment and, hence, consumer welfare in the (product) market(s) in which the firm operates. As an example, consider the case where $I = 100$, $r = 0.04$, $\mu_V = 0.06$, and $\sigma_V = 0.2$. That is, the project has a Sharpe ratio $h_V = 0.1$. Figure 5 shows the influence of h_S on the threshold V^* and on the probability of investment within 5 years for different initial values of V. As one can see, if one values relative

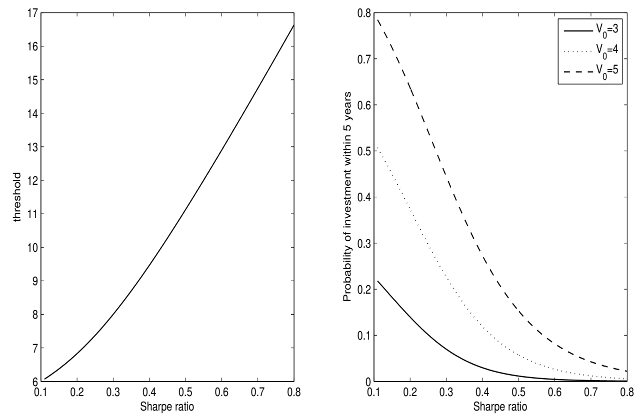

Figure 5: Comparative statics for h_S of the investment threshold (left panel) and the probability of investment within 5 years (right panel).

to an asset with a high Sharpe ratio the probability of investment within 5 years becomes negligible.

5 The Markets-Based Approach in Incomplete Markets

Recently, attention in the literature has been shifting to value projects in incomplete markets. Financial markets are incomplete if there does not exist a perfect hedge for the project's cash-flows. In other words, if there is idiosyncratic risk in the project, which is not traded on the financial market. In the market-based approach this means that the project's cash-flows and the prices of the risky asset are not perfectly correlated. Note that the issue of market completeness is irrelevant for the preference-based approach since in that method the discount factor and cash-flows are perfectly correlated by construction.

The most popular approach in the literature so far is to deal with market incompleteness (arising from the market-based approach) via a representative investor with a utility function exhibiting risk aversion (i.e. via the preference-based approach). See, for example, Van den Goorbergh et al. (2003), Hugonnier and Morellec (2005), and Miao and Wang (2005) for applications in this direction. In the preference-based approach, however, market incompleteness plays no role. Nevertheless, it is possible to apply the market-based approach, even though markets are incomplete, by, for example, bounding the value of the project in incomplete markets with so-called *good-deals bounds*. These are analysed in Cochrane and Saá-Requejo (2000), but go back to, at least, Ross (1976) (see also Ross (2005)). The idea is to find a maximum and minimum value for the project, while the volatility of the discount factor is exogenously bounded.

To be more precise, let the cash-flows of the project follow the diffusion

$$\frac{dV_t}{V_t} = \mu_V dt + \sigma_{V_z} dz + \sigma_{V_w} dw, \tag{14}$$

where $\mathbb{E}(dzdw) = 0$. For simplicity, it is assumed that the firm wishes to value the project relative to the riskless bond and the risky asset with price processes as given in (13). For further reference, let $\sigma_V^2 \equiv \sigma_{V_z}^2 + \sigma_{V_w}^2$ denote the total instantaneous variance of V, and let $h_V \equiv \frac{\mu_V - r}{\sigma_V}$, denote the Sharpe ratio of V. Note that the instantaneous correlation between V and S equals

$$\rho \equiv corr\Big(\frac{dV}{V}, \frac{dS}{S}\Big) = \frac{\sigma_{V_z}}{\sigma_V}.$$

Note that in the complete markets case above, it holds that $\rho = 1$, whereas for incomplete markets we have $\rho < 1$.

For every $\nu \in \mathbb{R}$, the discount factor

$$\frac{d\Lambda}{\Lambda} = -rdt - h_S dz - \nu dw,$$

prices the bond and riskless asset[7] and can, therefore, be used to value the project. The choice of ν, however, is arbitrary; market data alone are not enough.[8] For any $\nu \in \mathbb{R}$, we have (using Ito's lemma)

$$\frac{d\Lambda V}{\Lambda V} = -\sigma_V(\rho h_S + \nu\sqrt{1-\rho^2} - h_V)dt + (\sigma_{V_z} - h_S)dz + (\sigma_{V_w} - \nu)dw.$$

Let $\delta_\nu = \sigma_V(\rho h_S + \nu\sqrt{1-\rho^2} - h_V)$ be the convenience yield. From Proposition 1 it then follows that investment should take place as soon as the threshold $V_\nu^* = \frac{\beta_1}{\beta_1 - 1}\delta_\nu I$ is reached, where β_1 is the positive root of the quadratic equation

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_V^2\beta(\beta-1) + (r - \delta_\nu)\beta - r = 0.$$

Note that, for this result to hold we need to assume that $\delta > 0 \iff \rho h_S + \sqrt{1-\rho^2}\nu > h_V$. In the remainder it is assumed that this assumption holds.

We now proceed by bounding the instantaneous volatility of the discount factor, which equals

$$\mathbb{E}\Big(\frac{d\Lambda}{\Lambda}\Big)^2 = h_S^2 + \nu^2,$$

by a constant, say, $A^2 > h_S^2$. One can then find the maximum and the minimum value of the project under this volatility constraint. That is, one solves

$$\begin{aligned} \max_{\nu \in \mathbf{R}} \quad & F(V_t) = \frac{V_t}{\sigma_V(\rho h_S + \nu\sqrt{1-\rho^2} - h_V)} - I \\ \text{such that} \quad & h_S^2 + \nu^2 \leq A^2. \end{aligned}$$

Since the objective function is monotonically decreasing in ν, this problem is solved when

$$\nu = \bar{\nu} \equiv -\sqrt{A^2 - h_S^2} = -h_S\sqrt{\frac{A^2}{h_S^2} - 1}.$$

The investment threshold then equals $V_{\delta_{\bar{\nu}}}^*$, where

$$\delta_{\bar{\nu}} = \sigma_v\Big[h_S\Big(\rho - \sqrt{\tfrac{A^2}{h_S^2} - 1}\sqrt{1-\rho^2}\Big) - h_V\Big].$$

Similarly one can find the minimum value of the project under the volatility constraint. This leads to $\nu = \underline{\nu} \equiv +h_S\sqrt{\frac{A^2}{h_S^2} - 1}$, and investment threshold $V_{\delta_{\underline{\nu}}}^*$, where

$$\delta_{\underline{\nu}} = \sigma_v\Big[h_S\Big(\rho + \sqrt{\tfrac{A^2}{h_S^2} - 1}\sqrt{1-\rho^2}\Big) - h_V\Big].$$

[7]As long as Λ_0, the process is positive and, therefore, a discount factor.

[8]This is equivalent to non-uniqueness of the equivalent martingale measure (see, for example, Dana and Jeanblanc (2003)).

Note that $\underline{\nu}$ and $\bar{\nu}$ are the maximum and minimum loadings on dw, respectively, consistent with the volatility constraint. For $\rho = 1$, we are back in the familiar complete markets case and $V^*_{\delta_{\underline{\nu}}} = V^*_{\delta_{\bar{\nu}}}$.

As an illustration, consider the case with $I = 100$, and parameter values as in Table 3. For $A = \frac{3}{2}h_S$ and $A = 2h_S$, Figure 6 shows the bounds for $\rho \in [0.85, 1]$.

$r = 0.03$	$\mu_V = 0.06$
$\sigma_V = 0.4$	$h_S = 0.5$

Table 3: Parameter values

The bounds are only well-defined if $\delta_{\bar{\nu}} > 0$, which requires a high enough value for ρ. Note, furthermore, that the lower bounds correspond to $\bar{\nu}$, and the upper bounds to $\underline{\nu}$. As expected, the bounds are wider for lower values of ρ and higher values of

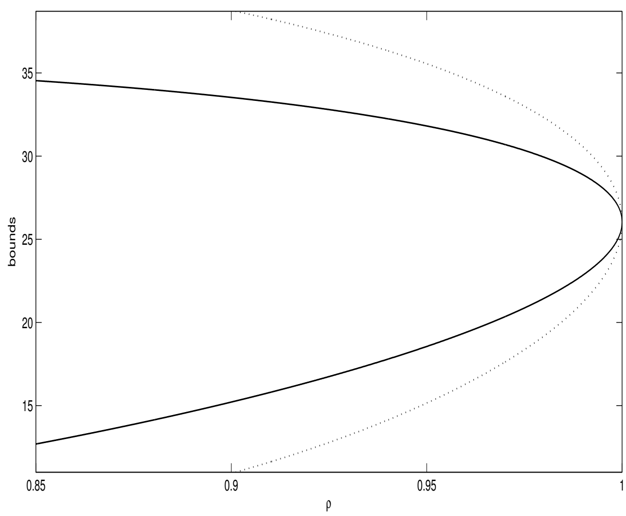

Figure 6: Good deal bounds for $A = \frac{3}{2}h_S$ (solid line) and $A = 2h_S$ (dashed line).

A.

It should be noted that the bounds provide a guideline for investment policy. The data obtained from financial markets are simply not precise enough to provide a clear-cut answer to the investment timing problem. One could choose to abandon the market-based approach completely in such cases and resort to the preference-based approach. Alternatively, the good-deal bounds are a good alternative to stretch the market-based approach a bit further.

Finally, it could be argued that financial markets are, at least, near complete and that it is only transaction costs, which prevent full market completeness. This issue, however, is irrelevant, since it only matters what assets the firm uses to evaluate the investment project.[9] It is unlikely that a firm will use all traded assets in such an analysis. It will most likely use a market index and maybe a few other assets. In such cases markets are very likely to be incomplete. For example, in most industries, the profitability of an investment depends on the investment behaviour of competitors. This creates a source of risk, which is unlikely to be traded on financial markets. Furthermore, a government's fiscal or environmental policy may influence a project's profitability. The risk this induces is typically not traded either.

6 Discussion

In this paper, a general approach to investment appraisal has been developed, based on a discount factor. In short, the discount factor represents how the decision maker interprets "making decisions in the interest of shareholders". A distinction is made between two fundamentally different approaches, the preference-based and the markets-based, respectively. It is illustrated that many different outcomes of ROT models in the literature are due to different choices of the discount factor. This suggests that ROT is a robust framework, where the main choice for the decision-maker is the choice of discount factor.

The preference-based approach could, in principle, be extended to a wide variety of utility functions. These need not necessarily follow the axioms of Expected Utility Theory. In recent years, the paradigm of the rational, utility maximising agent or investor has seriously been contended (cf. Schleifer (2000)). Others argue, however, that neoclassical theory is certainly not dead. For example, Ross (2005, p. 66) argues that behavioural finance is being "more defined by what it doesn't like about neoclassical finance than what it has to offer as an alternative". This discussion can be avoided by relying on the market-based approach.

One advantage of the preference-based approach over the market-based one is that it also works in incomplete markets. This is the case, because one explicitly specifies the preferences of a representative investor over all conceivable income or consumption streams. However, one often uses asset price data to calibrate the parameters governing time and risk preferences of investors. Furthermore, in incomplete markets one typically needs to value an income stream, which is not perfectly correlated to traded assets. Still one uses the same price data to obtain estimates of preference parameters of which one has no indication regarding their out-of-sample performance.

It seems that the second argument points to exactly the same problem that

[9] See Ross (2005) for an elaboration of this point.

arises in applying the market-based approach: One has to use data obtained from a given sample to make out-of-sample predictions. In the market-based approach one uses price data which do not cover all the risk incorporated in the project on hand. Bounding the project's value by imposing a "reasonable" bound on the variance of the discount factor seems, currently, the best one can do. Obviously, the exogenous volatility bound is arbitrary, but not more arbitrary than the assumption that in-sample preference estimates also hold out-of-sample.

One problem with ROT is that it is assumed that the firm can observe the project's cash-flows, even before investment has taken place. This is obviously not the case in many real-world applications. This might be a reason for the limited success of ROT in applied corporate finance. It would be interesting to extend the current model to situations where there is ambiguity concerning the actual cash-flow at any point before investment takes place. In an environment where the project is non-exclusive and competitors are present, this may lead to a strong second-mover advantage, which could delay investment. This, however, is left to future research.

References

Brealey, R.C. and S.C. Myers (2003). *Principles of Corporate Finance* (7^{th} ed.). McGraw–Hill, Boston.

Cochrane, J.H. (2005). *Asset Pricing* (Revised ed.). Princeton University Press, Princeton.

Cochrane, J.H. and J. Saá-Requejo (2000). Beyond Arbitrage: Good–Deal Asset Price Bounds in Incomplete Markets. *Journal of Political Economy*, **108**, 79–119.

Dana, R.-A. and M. Jeanblanc (2003). *Financial Markets in Continuous Time.* Springer–Verlag, Berlin.

Dixit, A.K. and R.S. Pindyck (1994). *Investment under Uncertainty.* Princeton University Press, Princeton.

Fudenberg, D. and J. Tirole (1985). Preemption and Rent Equalization in the Adoption of New Technology. *The Review of Economic Studies*, **52**, 383–401.

Goorbergh, R.W.J. van den, K.J.M. Huisman, and P.M. Kort (2003). Investment under Uncertainty and Risk Aversion in Incomplete Markets. CentER DP 2003-119, Tilburg University, Tilburg.

Grenadier, S.R. (2000). *Game Choices: The Intersection of Real Options and Game Theory.* Risk Books, London, UK.

Hendricks, K. , A. Weiss, and C. Wilson (1988). The War of Attrition in Continuous Time with Complete Information. *International Economic Review*, **29**, 663–680.

Hugonnier, J. and E. Morellec (2004). Investment under Uncertainty and Incomplete Markets. *Mimeo*, Université de Lausanne, Lausanne.

Hugonnier, J. and E. Morellec (2005). Real Options and Risk Aversion. *Mimeo*, Université de Lausanne, Lausanne.

Huisman, K.J.M. (2001). *Technology Investment: A Game Theoretic Real Options Approach*. Kluwer Academic Publishers, Dordrecht.

Huisman, K.J.M. , P.M. Kort, G. Pawlina, and J.J.J. Thijssen (2004). Strategic Investment under Uncertainty: Merging Real Options with Game Theory. *Zeitschrift für Betriebswissenschaft*, **67**, 97–123.

Lucas, R.E. (1978). Asset Prices in an Exchange Economy. *Econometrica*, **46**, 1429–1445.

Mas–Colell, A. , M.D. Whinston, and J.R. Green (1995). *Microeconomic Theory*. Oxford University Press, New York, NY.

McDonald, R. and D. Siegel (1986). The Value of Waiting to Invest. *Quarterly Journal of Economics*, **101**, 707–728.

Miao, J. and N. Wang (2005). Investment, Consumption, and Hedging under Incomplete Markets. *mimeo*, Boston University, Boston and Columbia Business School, New York.

Ross, S.A. (1976). The Arbitrage Theory of Asset Pricing. *Journal of Economic Theory*, **13**, 341–360.

Ross, S.A. (2005). *Neoclassical Finance*. Princeton University Press, Princeton.

Schleifer, A. (2000). *Inefficient Markets. An Introduction to Behavioral Finance*. Oxford University Press, Oxford.

Smets, F. (1991). Exporting versus FDI: The Effect of Uncertainty, Irreversibilities and Strategic Interactions. Working Paper, Yale University, New Haven.

Thijssen, J.J.J. , K.J.M. Huisman, and P.M. Kort (2006). The Effects of Information on Strategic Investment and Welfare. *Economic Theory*, **28**, 399–424.

Trigeorgis, L. (1996). *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. MIT Press, Cambridge, Mass.

Weeds, H.F. (2002). Strategic Delay in a Real Options Model of R&D Competition. *Review of Economic Studies*, **69**, 729–747.