Non-Exclusive Real Options and the Principle of Rent-Equalisation*

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Abstract

This paper considers the problem of the exercise decision of non-exclusive perpetual American-style real options in a two-player setting. Much of this literature has applied ideas from deterministic timing games directly to Real Options Theory, leading to unsatisfactory mathematical foundations. This paper presents a general model of non-exclusive real options. Existence of equilibrium is proved for the class of games with a first-mover advantage and where uncertainty is governed by a (possibly multi-dimensional) Markov process. The method is then applied to explicitly study equilibria in a two-player game where uncertainty is governed by a two-dimensional, correlated, geometric Brownian motion.

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JEL classification: C73, D81

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1 Introduction

It is common knowledge that games in continuous time are often difficult to analyse. Especially in so-called *preemption games*, i.e. situations where players have to decide on the time at which to take a certain action in the presence a first mover advantage. A historically interesting example would be a duel between two players walking away from each other with one bullet each. A more contemporary example considers two firms, both of which can invest in, for example, a new technology or a new product. The main reason for analytical difficulties in such games is that in continuous time the notion of the "time instant immediately after" a point in time is not well-defined (cf. Simon (1987a), Simon (1987b), and Simon and Stinchcombe (1989)).

In particular, it is difficult to model situations where players need to coordinate in a situation where it is optimal for one, and only one, player to take action.[1] For deterministic preemption games, Fudenberg and Tirole (1985) have presented a method to solve this problem by borrowing a technique from optimal control theory.[2] For two-player preemption games they show that there exists a symmetric mixed strategy equilibrium that involves *rent equalisation*. That is, in equilibrium the expected value of the first mover equals that of the second mover.

The concept of rent-equalisation has also been applied to game-theoretic extensions of real option models. The theory of real options (cf. Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit and Pindyck (1994)) deals with the problem of the optimal timing of investment decisions in the face of uncertainty. It views (real) investments as American-style call options and applies standard option pricing techniques (cf. Merton (1992), Musiela and Rutkowski (2005)) to value these real options and determine their optimal exercise time. The realisation that most real options are non-exclusive has led to the application of game theoretic tools to real option models.[3] The technique used in most applications is a direct, or simplified, application of the concepts of Fudenberg and Tirole (1985).[4]

By default, however, non-exclusive real option models deal with uncertainty and, hence, with stopping times, i.e. random variables, instead of deterministic time. Therefore, much of the analysis in Fudenberg and Tirole (1985) is not directly

[1] In their analysis of the Smets (1991) model, Dixit and Pindyck (1994, Chapter 9) refer to this in a footnote as "careful limiting considerations".

[2] Argenziano and Schmidt-Dengler (2006) study n-player preemption games, although they do not analyse the aforementioned coordination problem.

[3] See, for example, Smets (1991), Grenadier (2000), Huisman (2001), Huisman and Kort (1999), Weeds (2002), and Thijssen et al. (2006).

[4] In fact, in many cases the coordination problem is "solved" by simply assuming that players take action with equal probability and that joint action is impossible.

applicable to these models. A first step towards a formal analysis of game theoretic real option models is provided by Murto (2004), who considers exit in a duopoly with declining profitability. In that paper, however, the coordination problem does not arise and an equilibrium in pure strategies can be found. Murto (2004) uses the strategy concept introduced in Dutta and Rustichini (1995). In this framework, the profitability of each firm depends (deterministically) on how many firms are present in the industry and a random part, which follows a geometric Brownian motion (GBM). The firms then each choose a stopping set and exit as soon as the GBM hits their stopping set.

In this paper, the Dutta and Rustichini (1995) and Murto (2004) framework is extended in several ways. Firstly, I adapt the method of Fudenberg and Tirole (1985) to solve the coordination problem that often arises in preemption games. It is argued that the technique that is used in Fudenberg and Tirole (1985) can be formulated as a Markov chain in a suitably chosen state space. This makes the coordination device more intuitively appealing. Secondly, I prove existence of an equilibrium for general, possibly higher dimensional, Markov processes. This extends the standard literature, which is based on GBM. It is shown that the rent-equalisation principle also holds in this more general case. Finally, the equilibrium results are applied to a situation where two firms can invest in a project. The set-up is similar to Huisman (2001, Chapter 7), i.e. each firm's profits consists of a deterministic part, which depends on the number of firms having invested, and a random part. The novelty here is that each firm's profits is subject to different, but possibly correlated, GBMs. This introduces an asymmetry in an otherwise symmetric model. It is shown that there are three possible investment scenarios. Two in which one firm acts as an exogenously determined Stackelberg leader and the other firm as a Stackelberg follower, and one in which both firms try to preempt each other. In the latter case it holds that joint investment occurs with probability zero (like in most preemption models), but the probability with which each firm invests is not equal almost surely. This result indicates that one has to be careful with imposing exogenous assumptions on the solution to the coordination problem. As an example of a model with asymmetric uncertainty one can think of a situation with a domestic and a foreign firm, where the foreign firm has an additional source of risk, namely exchange rate risk.

The structure of the paper is as follows. In Section 2 the strategy and equilibrium concepts are presented. Existence of equilibrium is proved in Section 3. The application to asymmetric uncertainty in a duopoly is presented in Section 4 and Section 5 concludes.

2 Strategies and Equilibrium

This section introduces the equilibrium notions and accompanying strategy spaces used to analyse American-type real options. Throughout this section it is assumed that there are two players, indexed by $i \in \{1,2\}$. The two players each hold a perpetual option of either the call or the put type. The approach advocated here is a combination of the ideas presented in Fudenberg and Tirole (1985) for deterministic timing games and Dutta and Rustichini (1995) for a different class of stochastic games. The aim is to define an equilibrium that is the stochastic continuous time analogue of a subgame perfect Nash equilibrium. From a game theoretic point of view, the main problem in continuous time modelling is the absence of a well-defined notion of "immediately after time t" (cf. Simon and Stinchcombe (1989)). Dutta and Rustichini (1995) solve this problem by defining time as parameterised by two variables.

Definition 1 Time *is the two-dimensional set $\mathcal{T} = \mathbb{R}_+ \times \mathbb{Z}_+$, endowed with the lexicographic ordering and the standard topology induced by the lexicographic ordering.*

That is, a typical time element is a duplet $(t,s) \in \mathcal{T}$, which consists of a continuous and a discrete part. In the remainder, t refers to the continuous and s to the discrete component. The continuous component t can be thought of as "real time", whereas s represents "artificial time", used to model potential coordination problems between the players.

The underlying asset follows a strong Markov process, $(Y_t)_{t\geq 0}$, which is assumed to be a semimartingale[5], defined on a filtered probability space $\mathcal{P}_y = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P_y)$ and taking values in $(\mathbb{R}^d, \mathcal{B}^d)$, where $\mathcal{B}$ is the Borel σ-algebra. It is assumed, without loss of generality, that $(\Omega, \mathcal{F}) = (\mathbb{R}^{[0,\infty)}, \mathcal{B}^{[0,\infty)})$. The process starts at y, i.e. $P_y(Y_0 = y) = 1$, and the sample paths of Y are right-continuous and left-continuous over stopping times. The *sate of the game* is completely described by the process Y and whether or not each player has exercised the option. It is assumed that players make choices independently. Let $\Xi = \{0,1\}^2$ be the set of all possible combinations of the exercise status of the players. For each $t \geq 0$, the exercise status, X, changes according to a Markov chain on Ξ, $(X_s^t)_{s\in\mathbb{Z}_+}$, defined on a filtered probability space $\mathcal{P}_x = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_s)_{s\in\mathbb{Z}_+}, P_x)$, with $X_0^t = x$, P_x-a.s., as described below. For all $y \in \mathbb{R}_+$ and $x \in \Xi$, let $\mathcal{P}_{y,x}$ be the product space of $\mathcal{P}_y$ and $\mathcal{P}_x$, with probability measure $P_{y,x}$. For further reference, I define $X_t := \lim_{s\to\infty} X_s^t$.

Let $A \subset \mathbb{R}^d$. For further reference, define the stopping time

$$\tau_{y,x}(A) = \inf\{t \geq 0 | Y_t \in A, (Y_0, X_0) = (y,x), \quad P_y\text{-a.s.}\}.$$

[5]This includes, for example, Poisson processes, Brownian motion and Lévy processes.

Let $y \in \mathbb{R}_+$ and $x \in \Xi$. A strategy for Player i consists of a duplet $(S^i_{y,x}, \alpha^i_{y,x})$, where $S^i_{y,x} \subset \mathbb{R}_+$, and $\alpha^i_{y,x} \in (0,1]$. The set $S^i_{y,x}$ is referred to as Player i's *stopping set*, given the initial conditions (y,x). Note that it is not required, *a priori*, that the stopping set is connected. Let $\mathcal{S}^i_{y,x}$ denote the strategy space of Player i for initial state $(Y_0, X_0) = (y,x)$.

At $\tau_{y,x}(S^i_{y,x})$, the state of play changes in the following way. Player i starts playing a game in artificial time against Player j to determine who exercises the option at time $\tau_{y,x}(S^i_{y,x})$. In each round of this play, Player i exercises the option with probability $\alpha^i_{\tau_{y,x}(S^i_{y,x})}$, where for all $t \in \mathbb{R}_+$,

$$\alpha^i_t = \begin{cases} \alpha^i_{y,x} & \text{if } t = \tau_{y,x}(S^i_{y,x}) \\ 0 & \text{otherwise} \end{cases}$$

Play continues until *at least* one player exercises the option. Let $\tau_{y,x} = \tau_{y,x}(S^1_{y,x}) \wedge \tau_{y,x}(S^2_{y,x})$. For every $x \in \Xi$, this leads to a Markov chain $(X_s^{\tau_{y,x}(S^i_{y,x})})_{s \in \mathbb{Z}_+}$ on $\mathcal{P}_x$. For example, if $x = (0,0)$, this chain has transition probabilities that are given by

$$\begin{aligned} &P_x\big(X_s^{\tau_{y,x}} = (0,0) | X_{s-1}^{\tau_{y,x}} = (0,0)\big) = (1-\alpha^1_{\tau_{y,x}})(1-\alpha^2_{\tau_{y,x}}) \\ &P_x\big(X_s^{\tau_{y,x}} = (1,0) | X_{s-1}^{\tau_{y,x}} = (0,0)\big) = \alpha^1_{\tau_{y,x}}(1-\alpha^2_{\tau_{y,x}}) \\ &P_x\big(X_s^{\tau_{y,x}} = (0,1) | X_{s-1}^{\tau_{y,x}} = (0,0)\big) = (1-\alpha^1_{\tau_{y,x}})\alpha^2_{\tau_{y,x}} \\ &P_x\big(X_s^{\tau_{y,x}} = (1,1) | X_{s-1}^{\tau_{y,x}} = (0,0)\big) = \alpha^1_{\tau_{y,x}}\alpha^2_{\tau_{y,x}} \\ &P_x\big(X_s^{\tau_{y,x}} = X_{s-1}^{\tau_{y,x}} | X_s^{\tau_{y,x}} \in \{(1,0),(0,1),(1,1)\}\big) = 1. \end{aligned} \tag{1}$$

The chain has three absorbing states, $(1,0)$, $(0,1)$, and $(1,1)$. Note that, if α^i somehow depends on Y (as it typically will), then $\mathcal{P}_y$ and $\mathcal{P}_x$ are not independent. From these transition probabilities it follows that

$$\begin{aligned} p_{10} &\equiv P\big(X_{\tau_{y,x}} = (1,0)\big) = \frac{\alpha^1_{\tau_{y,x}}(1-\alpha^2_{\tau_{y,x}})}{\alpha^1_{\tau_{y,x}} + \alpha^2_{\tau_{y,x}} - \alpha^1_{\tau_{y,x}}\alpha^2_{\tau_{y,x}}} \\ p_{01} &\equiv P\big(X_{\tau_{y,x}} = (0,1)\big) = \frac{(1-\alpha^1_{\tau_{y,x}})\alpha^2_{\tau_{y,x}}}{\alpha^1_{\tau_{y,x}} + \alpha^2_{\tau_{y,x}} - \alpha^1_{\tau_{y,x}}\alpha^2_{\tau_{y,x}}} \\ p_{11} &\equiv P\big(X_{\tau_{y,x}} = (1,1)\big) = \frac{\alpha^1_{\tau_{y,x}}\alpha^2_{\tau_{y,x}}}{\alpha^1_{\tau_{y,x}} + \alpha^2_{\tau_{y,x}} - \alpha^1_{\tau_{y,x}}\alpha^2_{\tau_{y,x}}}. \end{aligned} \tag{2}$$

Note that $P_x(X_s^{\tau_{y,x}} \in \{(1,0),(0,1),(1,1)\}, s < \infty) = 1$, so that play in imaginary time is finite, P_x-a.s. In fact, this construction turns the coordination device into a game in strategic form with two pure strategies for each player, namely *exercise* and *don't exercise.* The mixed strategies over these pure strategies are $(\alpha, 1-\alpha)$, and the expected payoffs follow from the probability measure in (2).

For $x \in \{(1,0),(0,1)\}$, one player, say Player i, has already exercised the option. This implies that $\tau_{y,x} = \tau_{y,x}(S^j_{y,x})$, and, consequently, that $\alpha^i_t = 0$, for all $t \in \mathbb{R}_+$. The resulting Markov chain has only one absorbing state, namely $(1,1)$, which is reached in finite time.

Definition 2 *A* Markov strategy *for Player i, $i = 1,2$, specifies for all $y \in \mathbb{R}_+$ and all $x \in \Xi$ a duplet $\sigma^i_{y,x} = (S^i_{y,x}, \alpha^i_{y,x})$, where $S^i_{y,x} \subset \mathbb{R}_+$ and $\alpha^i_{y,x} \in (0,1]$, such that*

1. *$S^i_{y,x} = \emptyset$, if $x_i = 1$;*
2. *$\alpha^i_{y,x} = 1$, if $x_j = 1$.*

The two conditions are merely regularity conditions. The former ensures that a player can exercise the option only once. The latter condition is a convention. The possibility of $\alpha < 1$ is needed to allow for coordination between the players. If the Player j has already exercised, then Player i faces a decision-theoretic problem and there is no need for coordination. It is assumed that $\tau_{y,x}(S^i_{y,x}) = \infty$, P_y-a.s., if $S^i_{y,x} = \emptyset$.

The set of Markov strategies for player i is denoted by Σ^i. Given the state of the stochastic processes $(Y_t, X_t)_{t \in \mathbf{R}_+}$ the *instantaneous payoff* to Player i at time t is given by the mapping $V^i_{X_t} : Y_t \to \mathbb{R}$, which is assumed to be strictly increasing in Y. Note that the instantaneous payoffs are assumed to be Markovian in the sense that they only depend on the current state of the processes Y and X. If Player i exercises the option she incurs a sunk cost $I^i > 0$. It is assumed that players discount payoffs according to a discount factor $\left(\Lambda^i_t\right)_{t \geq 0}$, which is adapted to $\mathcal{P}$ and perfectly correlated with $(Y_t)_{t \geq 0}$.

Definition 3 *A two-player* non-exclusive real option game *(NERO) is a collection $\Gamma = \Big((Y_t)_{t\geq 0}, \big(\mathcal{S}^i, V^i, I^i, \left(\Lambda^i_t\right)_{t\geq 0}\big)_{i=1,2}\Big)$, with Y_0 given and $X^0_0 = (0,0)$.*

For further reference, we define the following discounted payoff functions. The *follower value*, $F^i(Y_t)$, for Player i is defined to be the value of the optimal stopping problem, given that the other player takes action when the value of the process $(Y_t)_{t\geq 0}$ equals y. That is,

$$
\begin{aligned}
F^i(y) &= \sup_{\tau \in \mathcal{M}} \mathbb{E}_y \Big[\int_0^\tau \Lambda^i_t V^i_{01}(Y_t) dt + \int_\tau^\infty \Lambda^i_t V^i_{11}(Y_t) dt - \Lambda^i_\tau I^i \Big] \\
&= \sup_{\tau \in \mathcal{M}} \mathbb{E}_y \Big[\int_0^\tau \Lambda^i_t V^i_{01}(Y_t) dt + \Lambda^i_\tau \mathbb{E}_{Y_\tau} \Big(\int_0^\infty \Lambda^i_t V^i_{11}(Y_t) dt - I^i \Big) \Big].
\end{aligned}
\tag{3}
$$

where $\mathcal{M}$ is the set of Markov times adapted to $\mathcal{P}_y$. Let S^i_F denote the optimal stopping set of (3).

The *leader value*, $L^i(y)$, for Player i is the expected discounted payoff stream if Player i exercises the option when the process $(Y_t)_{t\geq 0}$ has the value y, given that Player j exercises the option at time $\tau_y(S_j^F)$. That is,

$$L^i(y) = \mathbb{E}_y\Big[\int_0^{\tau_y(S_F^j)} \Lambda_t^i V_{10}^i(Y_t)dt + \int_{\tau_y(S_F^j)}^{\infty} \Lambda_t^i V_{11}^i(Y_t)dt - I^i\Big]. \tag{4}$$

Let $S_P^i = \{y \in \mathbb{R}^d | L^i(y) \geq F^i(y)\}$. Let S_L^i be the optimal stopping set of the problem

$$\begin{aligned}\bar{L}^i(y) &= \sup_{\tau\in\mathcal{M}} \mathbb{E}_y\Big[\int_0^{\tau} \Lambda_t^i V_{00}^i(Y_t)dt + \int_{\tau}^{\tau(S_F^j)} \Lambda_t^i V_{10}^i(Y_t)dt \\ &\qquad + \int_{\tau(S_F^j)}^{\infty} \Lambda_t^i V_{11}(Y_t)dt - \Lambda_\tau^i I^i\Big] \\ &= \sup_{\tau\in\mathcal{M}} \mathbb{E}_y\Big[\int_0^{\tau} \Lambda_t^i V_{00}^i(Y_t)dt + \Lambda_\tau^i L^i(Y_\tau)\Big].\end{aligned} \tag{5}$$

The optimal stopping time τ is the time at which Player i would exercise the option, if she knew that Player j could not preempt her. The following assumption is made on the optimal stopping sets.

Assumption 1 *For all players i, the optimal stopping sets S_F^i and S_L^i are non-empty.*

Thirdly, let $M^i(y)$ denote the expected discounted value to Player i if both players exercise the option at the same time when the process $(Y_t)_{t\geq 0}$ takes the value y, i.e.

$$M^i(y) = \mathbb{E}_y\Big[\int_0^{\infty} \Lambda_t^i V_{11}^i(Y_t)dt - I^i\Big]. \tag{6}$$

Let $S_M^i = \{y \in \mathbb{R}^d | M^i(y) \geq F^i(y)\}$ be the set of payoffs where simultaneous investment has a higher expected discounted payoff than the follower role. Finally, let $S_N^i = (S_P^i \cup S_L^i \cup S_F^i)^c$.

For $x = (0,0)$, the *expected discounted payoff* of the strategies $(\sigma^1, \sigma^2) \in \Sigma^1 \times \Sigma^2$ to Player i, under $P_{y,x}$ is then equal to

$$\begin{aligned}V_{y,x}^i(\sigma_{y,x}^i, \sigma_{y,x}^j) = \mathbb{E}_{y,x}\Big[&\int_0^{\tau_{y,x}} \Lambda_t^i V_{00}^i(Y_t)dt \\ &+ \mathbb{1}_{\tau(S_{y,x}^i)<\tau(S_{y,x}^j)}\Big(\int_{\tau_{y,x}}^{\tau(S_{y,x}^j)} \Lambda_t^i V_{10}^i(Y_t)dt + \int_{\tau(S_{y,x}^j)}^{\infty} \Lambda_t^i V_{11}^i(Y_t)dt - \Lambda_{\tau(S_{y,x}^i)}^i I^i\Big) \\ &+ \mathbb{1}_{\tau(S_{y,x}^i)>\tau(S_{y,x}^j)}\Big(\int_{\tau_{y,x}}^{\tau(S_{y,x}^i)} \Lambda_t^i V_{01}^i(Y_t)dt + \int_{\tau(S_{y,x}^i)}^{\infty} \Lambda_t^i V_{11}^i(Y_t)dt - \Lambda_{\tau(S_{y,x}^i)}^i I^i\Big) \\ &+ \mathbb{1}_{\tau(S_{y,x}^i)=\tau(S_{y,x}^j)} W_{Y_{\tau_{y,x}}}^i(\alpha_{y,x}^i, \alpha_{y,x}^j),\end{aligned}$$

where

$$\begin{aligned} W^i_{y,x}(\sigma^i_{y,x},\sigma^j_{y,x}) = {} & p_{11}\mathbb{E}_y\Big[\int_0^\infty \Lambda^i_t V^i_{11}(Y_t)dt - I^i\Big] \\ & + p_{10}\mathbb{E}_y\Big[\int_0^{\tau(S^j_{y,(1,0)})} \Lambda^i_t V^i_{10}(Y_t)dt + \int_{\tau(S^j_{y,(1,0)})}^\infty \Lambda^i_t V^i_{11}(Y_t)dt - I^i\Big] \\ & + p_{01}\mathbb{E}_y\Big[\int_0^{\tau(S^i_{y,(0,1)})} \Lambda^i_t V^i_{01}(Y_t)dt + \int_{\tau(S^i_{y,(0,1)})}^\infty \Lambda^i_t V^i_{11}(Y_t)dt - Y_{\tau(S^i_{y,(0,1)})} I^i\Big]. \end{aligned}$$

For $x \in \{(1,0),(0,1),(1,1)\}$, the expected discounted payoffs are

$$\begin{aligned} V^i_{y,(1,0)}(\sigma^i,\sigma^j) = & \mathbb{E}_y\Big[\int_0^{\tau(S^j_{y,(1,0)})} \Lambda^i_t V^i_{10}(Y_t)dt \\ & + \int_{\tau(S^j_{y,(1,0)})}^\infty \Lambda^i_t V^i_{11}(Y_t)dt - I^i\Big] \\ V^i_{y,(0,1)}(\sigma^i,\sigma^j) = & \mathbb{E}_y\Big[\int_0^{\tau(S^i_{y,(0,1)})} \Lambda^i_t V^i_{01}(Y_t)dt \\ & + \int_{\tau(S^i_{y,(0,1)})}^\infty \Lambda^i_t V^i_{11}(Y_t)dt - Y_{\tau(S^i_{y,(0,1)})} I^i\Big] \\ V^i_{y,(1,1)}(\sigma^i,\sigma^j) = & \mathbb{E}_y\Big[\int_0^\infty \Lambda^i_t V^i_{11}(Y_t)dt - I^i\Big], \end{aligned}$$

respectively.

A subgame perfect equilibrium is then defined as follows.

Definition 4 *Let Γ be a two-player NERO. A duplet of strategies $(\bar\sigma^1,\bar\sigma^2) \in \Sigma^1 \times \Sigma^2$ constitutes a* subgame perfect equilibrium *(SPE) if it prescribes a Nash equilibrium for all $(y,x) \in \mathbb{R}_+ \times \Xi$, i.e.*

$$\forall_{i\in\{1,2\}}\forall_{\sigma^i\in\Sigma^i}\forall_{(y,x)\in\mathbb{R}_+\times\Xi} : V^i_{y,x}(\bar\sigma^i,\bar\sigma^j) \geq V^i_{y,x}(\sigma^i,\bar\sigma^j), \quad P_y\text{-a.s.}$$

Note that in standard extensive form games, the notion of subgame perfectness is defined over time. Due to the strong Markov property of $(Y_t)_{t\geq 0}$, the definition of SPE above could equivalently be defined over stopping times.

3 Options with a First Mover Advantage

In this section American-type perpetual non-exclusive call options are studied. The following assumptions are made with respect to the instantaneous payoff functions and the optimal stopping sets.

Assumption 2 *For every Player i, $i \in \{1,2\}$, it holds that*

1. $V_{kl}^i : \mathbb{R}^d \to \mathbb{R}$ *is continuous for all* $k, l = 0, 1$*,*

2. $V_{10}^i(y) \geq V_{11}^i(y) > V_{00}^i(y) \geq V_{01}^i(y)$*, for all* $y \in \mathbb{R}^d$*,*

3. $S_F^i \subset S_L^i$*.*

The second condition ensures that exercising the option is always profitable if there were no sunk costs. The third condition ensures that, for each player, there are values for y, where she wants to be leader rather than follower. In other words, there is a first mover advantage.

Let
$$\Theta = (S_F^1 \cap S_F^2) \cup (S_L^1 \cap S_N^2) \cup (S_N^1 \cap S_L^2) \cup (S_P^1 \cap S_P^2),$$
and
$$\varphi^i(y) = \frac{L^i(y) - F^i(y)}{L^i(y) - M^i(y)},$$
for all y, such that $L^i(y) \neq M^i(y)$. Theorem 1 shows that at least one player exercises the option when the set Θ is entered for the first time.

Theorem 1 *Let G be a two-player NERO satisfying Assumptions 1 and 2. Let $y \in \mathbb{R}_+^d$ and $x \in \Xi$. Then $\bar{\sigma} = (\bar{\sigma}_{y,x}^1, \bar{\sigma}_{y,x}^2)_{(y,x)\in\mathbb{R}^d\times\Xi} \in \mathcal{S}^1 \times \mathcal{S}^2$, with $\bar{\sigma}_{y,x}^i = (\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i)$ constitutes a SPE, where*

$$(\bar{S}^i{}_{y,x}, \bar{\alpha}_{y,x}^i) = \begin{cases} (S_F^i, 1) & \text{if } \tau_{y,x}(\Theta) = \tau_{y,x}\big((S_F^1 \cap S_F^2) \\ & \quad \cup (S_N^i \cap S_L^j)\big),\ P_y\text{-a.s.} \\ (S_L^i, 1) & \text{if } \tau_{y,x}(\Theta) = \tau_{y,x}(S_L^i \cap S_N^j),\ P_y\text{-a.s.} \\ \big(S_P^i, \varphi^j(Y_{\tau_{y,x}(\Theta)})\big) & \text{if } \tau_{y,x}(\Theta) = \tau_{y,x}(S_P^1 \cap S_P^2),\ P_y\text{-a.s.}, \end{cases}$$

if $x = (0, 0)$,

$$(\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = (S_F^i, 1),$$

if $x = (0, 1)$, and

$$(\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = (\emptyset, 0),$$

otherwise.

Proof. Let $y \in \mathbb{R}^d$. First, consider the case where $x^j = 1$ and $x^i = 0$. Then, by definition, the optimal stopping set for Player i is S_F^i. So, $\bar{\sigma}_{y,x}^i$ is a dominant strategy.

Let $x=(0,0)$ and for all $i \in \{1,2\}$, let $\bar{S}^i_L = S^i_L \cap (S^i_F)^c$ and $\bar{S}^i_P = S^i_P \cap (S^i_L)^c$. First, note that, for all $y \in \mathbb{R}^d$,

$$\begin{aligned} M^i(y) &= \mathbb{E}_y\Big[\int_0^\infty \Lambda^i_t V^i_{11}(Y_t)dt - I^i\Big] \\ &\le \sup_{\tau \in \mathcal{M}} \mathbb{E}_y\Big[\int_0^\tau \Lambda^i_t V^i_{01}(Y_t)dt + \int_\tau^\infty \Lambda^i_t V^i_{11}(Y_t)dt - \Lambda^i_\tau I^i\Big] \\ &= F^i(y), \end{aligned}$$

since Player i can always choose $\tau = 0$, P_y-a.s. Hence, $S^i_M \subset S^i_F$ and $S^i_M = \{y \in \mathbb{R}^d | M^i(y) = F^i(y)\}$. Consider the following cases.

1. $y \in S^1_F \cap S^2_F$
Note that

$$V^i_{y,x}(\bar{\sigma}^i_{y,x}, \bar{\sigma}^j_{y,x}) = M^i(y).$$

If Player i plays $\tilde{\sigma}^i_{y,x} \neq \bar{\sigma}^i_{y,x}$, then $\tau_{y,x}(\tilde{S}^i_{y,x}) \ge \tau_{y,x} = 0$, P_y-a.s. If the inequality is strict, Player j exercises immediately and the state of play changes to $x=(0,1)$. Therefore,

$$V^i_{y,x}(\tilde{\sigma}^i_{y,x}, \bar{\sigma}^j_{y,x}) = F^i(y) = M^i(y),$$

which implies that Player i has no incentive to deviate.

2. $y \in S^i_N \cap \bar{S}^j_L$
Under $\bar{\sigma}$, Player j exercises immediately and Player i then exercises at $\tau_{y,x}(S^F_i)$, which is a dominant strategy for $x=(0,1)$. The only deviation $\tilde{\sigma}^i_{y,x} = (\tilde{S}_{y,x}, \tilde{a}_{y,x})$ from $\bar{\sigma}^i_{y,x}$ which could potentially be profitable has $\tau_{y,x}(\tilde{S}_{y,x}) = 0$. In that case, we have

$$\begin{aligned} V^i_{y,x}(\tilde{\sigma}_{y,x}, \bar{\sigma}^j_{y,x}) =& W^i_{y,x}(\tilde{\sigma}_{y,x}, \bar{\sigma}^j_{y,x}) \\ =&(1-\tilde{\alpha}_0)F^i(y) + \tilde{\alpha}_0 M^i(y) \\ \le& F^i(y) = V^i_{y,x}(\bar{\sigma}^i_{y,x}, \bar{\sigma}^i_{y,x}). \end{aligned}$$

3. $y \in \bar{S}^i_L \cap S^j_N$
Note that $\bar{S}^i_L$ is a subset of the optimal stopping set of (5). Therefore, given that Player j exercises the option at $\tau_{y,x}(S^j_F)$, there is, by definition, no deviation that can yield a higher expected discounted payoff.

4. $y \in \bar{S}^1_P \cap \bar{S}^2_P$
In this case it holds that $\tau_{y,x} = 0$, P_y-a.s. The strategies $\bar{\sigma}$ lead to an expected payoff

$$V^i_{y,x}(\bar{\sigma}^i_{y,x}, \bar{\sigma}^j_{y,x}) = W^i_{y,x}(\bar{\sigma}^i_{y,x}, \bar{\sigma}^j_{y,x}) = F^i(y).$$

Let $\tilde{\sigma}^i \neq \bar{\sigma}^i$ be such that $\tau_{y,x}(\tilde{S}^i_{y,x}) \neq 0$. Then

$$V^i_{y,x}(\tilde{\sigma}^i_{y,x}, \bar{\sigma}^j_{y,x}) \leq V^i_{y,x}\big((S^i_F, 1), \bar{\sigma}^j\big) = F^i(y).$$

On the other hand, if $\tilde{\sigma}^i_{y,x}$ is such that $\tau_{y,x}(\tilde{S}^i_{y,x}) = 0$, the players are effectively playing the game depicted in Figure 1. The cell (*continue*, *continue*) has no pay-

	Exercise	Continue
Exercise	$M^i(y), M^j(y)$	$L^i(y), F^j(y)$
Continue	$F^i(y), L^j(y)$	

Figure 1: The coordination game.

off, since this outcome is impossible under the Markov chain (1). When $\sigma^i_{y,x} = (S^i_{y,x}, \alpha^i_{y,x})$ and $\sigma^j_{y,x} = (S^j_{y,x}, \alpha^j_{y,x})$ are such that $\tau_{y,x}(S^i_{y,x}) = \tau_{y,x}(S^j_{y,x})$, P_y-a.s., then $V^i_{y,x}(\sigma^i_{y,x}, \sigma^j_{y,x}) = W^i_{y,x}(\sigma^i, \sigma^j)$. Note that

$$W^i_{y,x}\big((S^i_{y,x}, 1), (S^j_{y,x}, \alpha^j_{y,x})\big) = (1 - \alpha^j_{y,x})L^i(y) + \alpha^j_{y,x}M^i(y),$$

and

$$W^i_{y,x}\big((S^i_{y,x}, 0), (S^j_{y,x}, \alpha^j_{y,x})\big) = F^i(y),$$

for all $\alpha^j_{y,x} \in [0,1]$. We have that $W^i_{y,x}\big((S^i_{y,x}, 1), (S^j_{y,x}, \alpha^j_{y,x})\big) > W^i_{y,x}\big((S^i_{y,x}, 0), (S^j_{y,x}, \alpha^j_{y,x})\big) \iff \alpha^j_{y,x} < \varphi^i(y)$. In other words, the best-response functions, $\big(B^i(\alpha^j_{y,x}), B^j(\alpha^i_{y,x})\big)$, for both players can be depicted as in Figure 2. The point $\big(\varphi^j(y), \varphi^i(y)\big)$ is the only

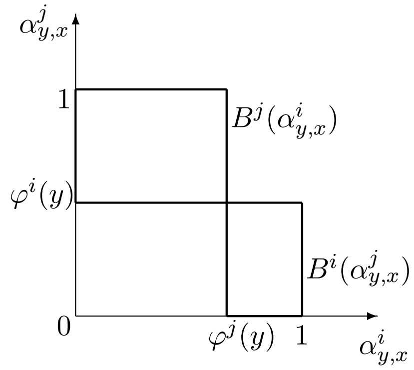

Figure 2: Best response functions.

mixed strategy Nash equilibrium[6] and, hence, unilateral deviations do not lead to higher expected utility.

[6]Note that there are two pure-strategy equilibria as well; one where Player i becomes leader and Player j follower with probability one, and the symmetric counter-part.

5. $y \in S_N^1 \cap S_N^2$
Note that, in this case $\tau_{y,x}$ is the first hitting time of Θ. So, what remains to be shown is that, for all elements of Θ, waiting until Θ is entered is an equilibrium. Under $\bar{\sigma}$, the expected payoff to Player i equals

$$V_{y,x}^i(\bar{\sigma}_{y,x}^i, \bar{\sigma}_{y,x}^j) = \mathbb{E}_y\Big[\int_0^{\tau_{y,x}} \Lambda_t^i V_{00} dt + \Lambda_{\tau_{y,x}}^i V_{Y_{\tau_{y,x}},x}^i(\bar{\sigma}_{y,x}^i, \bar{\sigma}_{y,x}^j)\Big]$$
$$= \begin{cases} \mathbb{E}_y\Big[\int_0^{\tau_{y,x}} \Lambda_t^i V_{00} dt + \Lambda_{\tau_{y,x}}^i L^i(Y_{\tau_{y,x}})\Big] & \text{if } \tau_{y,x} = \tau_{y,x}(S_L^i \cap S_N^j), \\ & P_y\text{-a.s.} \\ \mathbb{E}_y\Big[\int_0^{\tau_{y,x}} \Lambda_t^i V_{00} dt + \Lambda_{\tau_{y,x}}^i F^i(Y_{\tau_{y,x}})\Big] & \text{otherwise.} \end{cases}$$

The only deviation $\tilde{\sigma}^i$ of $\bar{\sigma}^i$ that could possibly lead to a higher payoff has $\tilde{S}_{y,x}^i$, such that $\tilde{S}_{y,x}^i \cap S_N^i \neq \emptyset$ and $\tau_{y,x}(\tilde{S_{y,x}}^i) < \tau_{y,x}(\Theta)$, P_y-a.s. But, by definition, $\tau_{y,x}(\tilde{S}_{y,x}^i)$ does not solve (5). Hence, waiting to exercise the option is weakly dominant. $\square$

Note that in the region $S_P^1 \cap S_P^2$, it is not optimal for either player to exercise the option. Along every equilibrium path, however, at least one player exercises as soon as this set is hit. Therefore, this set is called the *preemption* set. Note that, in equilibrium, expected payoffs in the preemption set are equal to the follower payoffs for both players. This phenomenon is called *rent-equalisation.*

As a corollary, suppose that firms are symmetric, i.e. $V_{kl}^1 = V_{kl}^2 \equiv V_{kl}$, all $k, l \in \{0,1\}$, and $I^1 = I^2 \equiv I$. In that case all stopping regions are the same and the following result follows immediately from Theorem 1.

Corollary 1 *Let G be a symmetric two-player NERO satisfying Assumptions 1 and 2. Let $y \in \mathbb{R}_+^d$ and $x \in \Xi$. Then $\bar{\sigma} = (\bar{\sigma}_{y,x}^1, \bar{\sigma}_{y,x}^2)_{(y,x)\in\mathbf{R}^d\times\Xi} \in \mathcal{S}^1 \times \mathcal{S}^2$, with $\bar{\sigma}_{y,x}^i = (\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i)$ constitutes a SPE, where*

$$(\bar{S^i}_{y,x}, \bar{\alpha}_{y,x}^i) = \begin{cases} (S_F, 1) & \text{if } \tau_{y,x}(\Theta) = \tau_{y,x}(S_F),\ P_y\text{-a.s.} \\ \big(S_P, \varphi(Y_{\tau_{y,x}(\Theta)})\big) & \text{if } \tau_{y,x}(\Theta) = \tau_{y,x}(S_P),\ P_y\text{-a.s.}, \end{cases}$$

if $x = (0,0)$,

$$(\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = (S_F, 1),$$

if $x = (0,1)$, and

$$(\bar{S}_{y,x}^i, \bar{\alpha}_{y,x}^i) = (\emptyset, 0),$$

otherwise.

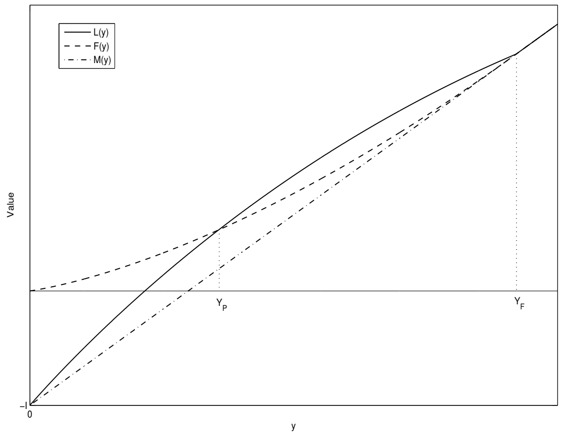

Figure 3: Payoff functions for a symmetric NERO with a one-dimensional Markov process.

In the case that $(Y_t)_{t \geq 0}$ is one-dimensional, a typical plot of the payoff functions is given in Figure 3, where $S^F = [Y_F, \infty)$, $S_N = [0, Y_P)$, and $S_P = [Y_P, Y_F)$. If $y \in S_N$, $x = (0,0)$, and $(Y_t)_{t \geq 0}$ has continuous sample paths, then $\tau_{y,x} = \tau_{y,x}(S^P)$, P_y-a.s. and each player exercises the option at time $\tau_{y,x}$ with probability $p_{10} = p_{01} = \frac{1}{2}$, since

$$\alpha_{\tau_{y,x}} = \frac{L(Y_P) - F(Y_P)}{L(Y_P) - M(Y_P)} = 0.$$

Note that if $(Y_t)_{t \geq 0}$ exhibits jumps, there can be a positive probability that both players exercise simultaneously in the preemption set.

4 An Example with Asymmetric Uncertainty

Consider the following setting where the payoff stream of Player i equals

$$u^i(Y, X) = \Lambda^i D_{kl} Y_i,$$

for $k, l = 0, 1$ and uncertainty is driven by a two-dimensional geometric Brownian motion (GBM),

$$\frac{dY}{Y} = \mu_Y dt + \Sigma_Y dz,$$

where $\mu = (\mu_1, \mu_2) \geq 0$ is the trend, $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is the matrix of instantaneous volatilities, and $dz = (dz_1, dz_2)$, where z_1 and z_2 are independent Wiener processes. Let $\sigma_k^2 = \sigma_{k1}^2 + \sigma_{k2}^2$, $k = 1, 2$, be the total instantaneous variance of Y_k. The instantaneous correlation between Y_1 and Y_2 is

$$\rho = \frac{1}{dt}\mathbb{E}\Big(\frac{dY_1}{Y_1}\frac{dY_2}{Y_2}\Big) = \frac{\sigma_{11}\sigma_{21}+\sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}.$$

The discount factor for Player i is assumed to be equal to

$$\frac{d\Lambda^i}{\Lambda^i} = -\mu_{\Lambda^i}dt - \sigma_{\Lambda^i_1}dz_1 - \sigma_{\Lambda^i_2}dz_2.$$

The instantaneous payoffs are taken to be linear in Y_i, so $V^i_{kl}(y_i) = D_{kl}y_i$, for all $y_i \in \mathbb{R}_+$, and $k, l \in \{0,1\}$. Note that the deterministic parts in the instantaneous payoffs are equal for both players. It is assumed that

1. $D_{10} > D_{11} > D_{00} \geq D_{01}$,

2. $D_{10} - D_{00} > D_{11} - D_{01}$.

These are standard assumptions to model a game with a first-mover advantage (cf. Huisman (2001)). Finally, sunk costs are the same for both players and equal to $I > 0$. That is, players are symmetric up to the uncertainty that underlies their payoffs, which, in turn, are correlated. For instance, one could think of a situation with a domestic and a foreign firm facing a similar domestic investment option. In such a case, both firms' profits depend on domestic uncertainty, z_1, whereas the foreign firm also incurs exchange rate risk, z_2.

The discounted profit stream to Player i, therefore, follows the SDE

$$\begin{aligned}\frac{d\Lambda^i Y_i}{\Lambda^i Y_i} =& \frac{dY_i}{Y_i} + \frac{d\Lambda^i}{\Lambda^i} + \frac{dY_i}{Y_i}\frac{d\Lambda^i}{\Lambda^i}\\ =& -(\mu_{\Lambda^i} + \sigma_{i1}\sigma_{\Lambda^i_1} + \sigma_{i2}\sigma_{\Lambda^i_2} - \mu_i)dt\\ &+ (\sigma_{i1} - \sigma_{\Lambda^i_1})dz_1 + (\sigma_{i2} - \sigma_{\Lambda^i_2})dz_2\\ \equiv& -\delta_{ii}dt + (\sigma_{i1} - \sigma_{\Lambda^i_1})dz_1 + (\sigma_{i2} - \sigma_{\Lambda^i_2})dz_2,\end{aligned} \tag{7}$$

where δ_{ii} is the convenience yield of Player i with respect to Y_i. Also define the convenience yield of Player i with respect to Y_j,

$$\delta_{ij} = \mu_{\Lambda^i} + \sigma_{\Lambda^i_1}\sigma_{j1} + \sigma_{\Lambda^i_2}\sigma_{j2} - \mu_j.$$

It is assumed throughout that $\delta_{ii} > 0$ and $\delta_{ij} > 0$. From (7) it follows that the value of simultaneous exercising of the option to Player i equals

$$M^i(Y_i) = \mathbb{E}_{Y_i}\Big[\int_0^\infty \Lambda^i_t D_{11} Y_{it} dt\Big] = \frac{D_{11}}{\delta_{ii}}Y_i - I.$$

4.1 Follower Value

First, I derive the follower value, $F^i(Y_i)$, for Player i. Note that this value does not depend on Y_j, since Player j has already exercised her option. The value of exercising the option – the option's "strike price" – at Y_i is $M^i(Y_i)$. Denote the value the option to exercise at Y_i by $C_F^i(Y_i)$. Then, the no-arbitrage value (relative to Λ^i) of $C_F^i(\cdot)$ should satisfy (cf. Cochrane (2005)),

$$\begin{aligned} &\Lambda^i D_{01} Y_i dt + \mathbb{E}_{Y_i}[d\Lambda^i C_F^i] = 0 \\ \iff & D_{01} Y_i dt + \mathbb{E}_{Y_i}[dC_F^i] + \mathbb{E}_{Y_i}\Big[C_F^i \frac{d\Lambda^i}{\Lambda^i}\Big] = -\mathbb{E}_{Y_i}\Big[\frac{d\Lambda^i}{\Lambda^i} dC_F^i\Big]. \end{aligned} \tag{8}$$

From Ito's lemma it follows that

$$\begin{aligned} dC_F^i =& \frac{\partial C_F^i}{\partial Y_i} dY_i + \frac{1}{2}\frac{\partial^2 C_F^i}{\partial Y_i^2} dY_i^2 \\ =& \Big(\frac{1}{2}\frac{\partial^2 C_F^i}{\partial Y_i^2}\sigma_i^2 Y_i^2 + \mu_i \frac{\partial C_F^i}{\partial Y_i} Y_i + D_{01} Y_i\Big) dt \\ &+ \frac{\partial C_F^i}{\partial Y_i} Y_i(\sigma_{i1} dz_1 + \sigma_{i2} dz_2). \end{aligned}$$

Substitution in (8) gives a second order PDE with general solution

$$C_F^i(Y_i) = A_F^i Y_i^{\beta_{ii}} + B_F^i Y_i^{\gamma_{ii}} + \frac{D_{01}}{\delta_{ii}} Y_i,$$

where A_F^i and B_F^i are constants and $\beta_{ii} > 1$ and $\gamma_{ii} < 0$ are the roots of the equation

$$\mathcal{Q}_{ii}(\xi) \equiv \frac{1}{2}\sigma_i^2 \xi(\xi - 1) + (\mu_{\Lambda^i} - \delta_{ii})\xi - \mu_{\Lambda^i} = 0.$$

Under the standard boundary condition, $\lim_{y_i \downarrow 0} C_F^i(y_i) = 0$, and the value-matching and smooth-pasting conditions (cf. Øksendal (2000) and Peskir and Shiryaev (2006)) it is obtained that $S_F^i = \{Y \in \mathbb{R}_+^2 | Y_i \in [Y_F^i, \infty)\}$, where

$$Y_F^i = \frac{\beta_{ii}}{\beta_{ii} - 1}\frac{\delta_{ii}}{D_{11} - D_{01}} I.$$

The follower value is then equal to

$$F^i(Y_i) = \begin{cases} \frac{1}{\beta_{ii}\delta_{ii}}(Y_F^i)^{1-\beta_{ii}} Y_i^{\beta_{ii}} + \frac{D_{01}}{\delta_{ii}} Y_i & \text{if } Y_i < Y_F^i \\ \frac{D_{11}}{\delta_{ii}} Y_i - I & \text{if } Y_i \geq Y_F^i. \end{cases}$$

4.2 Leader Value

Having established the value for the follower I now turn to the leader value. In deriving the leader value I assume that Player j cannot invest before Player i.

Therefore, the leader value for Player i can only be computed when $Y_j < Y_F^j$. If Player i becomes the leader, then, by definition, Player j becomes the follower. The exercise decision of Player j, which depends on Y_j, as we saw, influences the payoff to Player i. Hence, her leader value depends on Y_i and Y_j.

The no-arbitrage value of $L^i(Y_i, Y_j)$ follows[7]

$$D_{10} Y_i dt + \mathbb{E}_Y[dL^i] + \mathbb{E}_Y\Big[L^i \frac{d\Lambda^i}{\Lambda^i}\Big] = -\mathbb{E}_Y\Big[\frac{d\Lambda^i}{\Lambda^i} dL^i\Big]. \tag{9}$$

From Ito's lemma it follows that

$$\begin{aligned} dL^i =& L_i^i dY_i + L_j^i dY_j + \frac{1}{2} L_{ii}^i dY_i^2 + \frac{1}{2} L_{jj}^i dY_j^2 + L_{ij}^i dY_i dY_j \\ =& \big(\frac{1}{2}\sigma_i^2 Y_i^2 L_{ii}^i + \frac{1}{2}\sigma_j^2 Y_j^2 L_{jj}^i + \rho Y_i Y_j L_{ij}^i + \mu_i Y_i L_i^i + \mu_j Y_j L_j^i\big) dt \\ &+ (\sigma_{i1} Y_i L_i^i + \sigma_{j1} Y_j L_j^i) dz_1 + (\sigma_{i2} Y_i L_i^i + \sigma_{j2} Y_j L_j^i) dz_2 \end{aligned}$$

Substitution in (9) leads to a second order PDE with general solution

$$L^i(Y_i, Y_j) = A_{ii}^L Y_i^{\beta_{ii}} + B_{ii}^L Y_i^{\gamma_{ii}} + A_{ij}^L Y_j^{\beta_{ij}} + B_{ij}^L Y_j^{\gamma_{ij}} + \frac{D_{10}}{\delta_{ii}} Y_i - I,$$

where A_{ii}^L, $A_{i,j}^L$, B_{ii}^L, and B_{ij}^L are constants and $\beta_{ij} > 1$ and $\gamma_{ij} < 0$ are the roots of the quadratic equation

$$\mathcal{Q}_{ij}(\xi) \equiv \frac{1}{2}\sigma_j^2 \xi(\xi - 1) + (\mu_{\Lambda^i} - \delta_{ij})\xi - \mu_{\Lambda^i} = 0.$$

If $Y_j = 0$, then the threshold Y_F^j will never be reached and, hence, Player i will receive D_{10} over the time interval $[0, \infty)$. This leads to the boundary condition $\lim_{Y_j \downarrow 0} L^i(Y_i, Y_j) = \frac{D_{10}}{\delta_{ii}} Y_i - I$. This implies that $A_{ii}^L = B_{ij}^L = 0$. Also, if $Y_i = 0$, then Player i only incurs the sunk costs. This leads to the boundary condition $\lim_{Y_i \downarrow 0} L^i(Y_i, Y_j) = -I$, which implies $B_{ii}^L = 0$. Finally, if $Y_j = Y_F^j$, then both players exercise simultaneously. Therefore, another boundary condition is given by $L^i(Y_i, Y_F^j) = M^i(Y_i) = \frac{D_{11}}{\delta_{ii}} Y_i - I$. Solving for A_{ij} then gives

$$L^i(Y_i, Y_j) = \frac{D_{10}}{\delta_{ii}} Y_i - \frac{D_{10} - D_{11}}{\delta_{ii}} \Big(\frac{Y_j}{Y_F^j}\Big)^{\beta_{ij}} Y_i - I. \tag{10}$$

Note that the second term in (10) is a correction for the possibility that Player j may exercise her option as well at some time.

The value function in (10) is the strike price of the option to Player i of becoming the leader and can, therefore, be used to derive the optimal stopping set S_L^i in the following way. For $Y \in S_N^i$, let $C^i(Y_i, Y_j)$ denote the option value of Player i of

[7]For $f(x_1, x_2)$, let $f_i(\cdot) = \frac{\partial f(\cdot)}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j}$.

exercising the option with strike price governed by (10). The no-arbitrage value of $C^i(Y_i, Y_j)$ follows

$$D_{00}Y_i dt + \mathbb{E}_Y[dC^i] + \mathbb{E}_Y\Big[C^i \frac{d\Lambda^i}{\Lambda^i}\Big] = -\mathbb{E}_Y\Big[\frac{d\Lambda^i}{\Lambda^i} dC^i\Big]. \tag{11}$$

From Ito's lemma it follows that

$$\begin{aligned} dC^i =& C^i_i dY_i + C^i_j dY_j + \frac{1}{2} C^i_{ii} dY_i^2 + \frac{1}{2} C^i_{jj} dY_j^2 + C^i_{ij} dY_i dY_j \\ =& \big(\frac{1}{2}\sigma_i^2 Y_i^2 C^i_{ii} + \frac{1}{2}\sigma_j^2 Y_j^2 C^i_{jj} + \rho Y_i Y_j C^i_{ij} + \mu_i Y_i C^i_i + \mu_j Y_j C^i_j\big) dt \\ &+ (\sigma_{i1} Y_i C^i_i + \sigma_{j1} Y_j C^i_j) dz_1 + (\sigma_{i2} Y_i C^i_i + \sigma_{j2} Y_j C^i_j) dz_2 \end{aligned}$$

Substitution in (11) leads to a second order PDE with general solution

$$C^i(Y_i, Y_j) = A_{ii} Y_i^{\beta_{ii}} + B_{ii} Y_i^{\gamma_{ii}} + A_{ij} Y_j^{\beta_{ij}} + B_{ij} Y_j^{\gamma_{ij}} + \frac{D_{00}}{\delta_{ii}} Y_i - I,$$

where A_{ii}, $A_{i,j}$, B_{ii}, and B_{ij} are constants. If $Y = (0, 0)$, then the sets S^i_L, S^j_L, S^i_F, and S^j_F will never be reached and, hence, Player i will receive D_{00} over the time interval $[0, \infty)$. This leads to the boundary condition $\lim_{Y \downarrow (0,0)} C^i(Y_i, Y_j) = \frac{D_{00}}{\delta_{ii}} Y_i$. This implies that $B_{ii} = B_{ij} = 0$. The value-matching and smooth-pasting conditions, in this case, are

$$\begin{cases} C^i(Y_i, Y_j) = L^i(Y_i, Y_j) \\ \frac{\partial C^i(Y_i,Y_j)}{\partial Y_i} = \frac{\partial L^i(Y_i,Y_j)}{\partial Y_i} \\ \frac{\partial C^i(Y_i,Y_j)}{\partial Y_j} = \frac{\partial L^i(Y_i,Y_j)}{\partial Y_j} \end{cases}$$

Solving this system of equations leads to the optimal stopping set

$$S^i_L = \{Y \in \mathbb{R}^2_+ | Y_j \leq Y^j_F, Y_i \in [Y^i_L(Y_j), \infty)\}$$

where

$$Y^i_L(Y_j) = \frac{\beta_{ii}\delta_{ii}}{(\beta_{ii} - 1)(D_{10} - D_{00}) + (D_{10} - D_{11})(Y_j/Y^j_F)^{\beta_{ij}}} I.$$

The following lemma establishes the existence of a first-mover advantage.

Lemma 1 *For $i \in \{1, 2\}$ and $Y_j \leq Y^j_F$, it holds that $S^i_F \subset S^i_L(Y_j)$.*

Proof. Let $Y_j = Y^j_F$. Since $D_{10} \geq D_{11}$, and $D_{10} - D_{00} > D_{11} - D_{01}$ it immediately follows that

$$\begin{aligned} Y^i_L(Y^j_F) =& \frac{\beta_{ii}\delta_{ii}}{(\beta_{ii} - 1)(D_{10} - D_{00}) + (D_{10} - D_{00})} I \\ \leq& \frac{\beta_{ii}}{\beta_{ii} - 1} \frac{\delta_{ii}}{D_{10} - D_{00}} I \\ <& \frac{\beta_{ii}}{\beta_{ii} - 1} \frac{\delta_{ii}}{D_{11} - D_{01}} I = Y^i_F. \end{aligned}$$

Furthermore, it is easy to see that $\frac{\partial Y_L^i(Y_j)}{\partial Y_j} \leq 0$, for $Y_j < Y_F^j$. Hence, for all $Y_j \leq Y_F^j$, it holds that $Y_L^i(Y_j) < Y_F^i$. $\square$

For every $Y_j \leq Y_F^j$, let $Y_P^i(Y_j)$ be the solution of the equation $L^i(Y_P^i(Y_j), Y_j) = F^i(Y_P^i(Y_j))$. It is then easy to see that

$$S_P^i = \{Y \in \mathbb{R}_+^2 | Y_j \leq Y_F^j, Y_i \in [Y_P^i(Y_j), \infty)\}.$$

By construction of (10) it holds that $Y_P^i(Y_j) \leq Y_L^i(Y_j)$, for all $Y_j \leq Y_F^j$. Therefore, it holds that $S_P^i \subset S_L^i$. Let $\bar{S}_P^i = S_P^i \backslash S_L^i$ be the preemption region, as in the proof of Theorem 1. The following lemma establishes that this region is non-empty.

Lemma 2 *It holds that* $\bar{S}_P^1 \cap \bar{S}_P^2 \neq \emptyset$.

Proof. Let $\mathcal{A} = [0, Y_L^1(Y_F^2)] \times [0, Y_L^2(Y_F^1)]$ and define the function $f : \mathcal{A} \to \mathbb{R}^2$, where, for $i = 1, 2$, $f_i(y) = F^i(y_i) - L^i(y)$. Note that for $i = 1, 2$ and $Y_j \in [0, Y_L^2(Y_F^1)]$, by Lemma 1, it holds that

$$f_i(Y_L^i(Y_F^j), Y_j) < F^i(Y_F^i) - L^i(Y_F^i, Y^j) = 0 \tag{12}$$

$$f_i(0, Y_j) = I > 0 \tag{13}$$

Since $\mathcal{A}$ is a convex and compact set, and f is a continuous function, there exists a stationary point on $\mathcal{A}$ (cf. Eaves (1971)), i.e.

$$\exists_{y^* \in \mathcal{A}} \forall_{y \in \mathcal{A}} : y f(y^*) \leq y^* f(y^*). \tag{14}$$

Let $i \in \{1, 2\}$. Suppose that $y_i^* > 0$. Then there exists $\varepsilon > 0$, such that $y = y^* - \varepsilon e_i \in \mathcal{A}$, where e_i is the i-th unit vector. From (14) it then follows that

$$y f(y^*) - y^* f(y^*) = -\varepsilon f_i(y^*) \leq 0 \iff f_i(y^*) \geq 0. \tag{15}$$

Similarly, if $y_i^* < Y_L^i(Y_F^j)$, there exists $\varepsilon > 0$, such that $y = y^* + \varepsilon e_i \in \mathcal{A}$. Therefore,

$$y f(y^*) - y^* f(y^*) = \varepsilon f_i(y^*) \leq 0 \iff f_i(y^*) \leq 0. \tag{16}$$

Hence, from (15) and (16) it follows that $f(y^*) = 0$, if $y^* \in \mathcal{A} \backslash \partial\mathcal{A}$.

Suppose that $y_i^* = 0$, and let $y \in \mathcal{A}$ be such that $y_j = y_j^*$. Then (14) implies that $(y - y^*) f(y^*) = y_i f_i(y^*) \leq 0 \iff f_i(y^*) \leq 0$, which contradicts (13). Similarly, supposing that $y_i^* = Y_L^i(Y_F^j)$, and taking $y \in \mathcal{A}$ such that $y_i = y_i^*$, it is obtained that $(y - y^*) f(y^*) = (y_i - y_i^*) f_i(y^*) \leq 0 \iff f_i(y^*) \geq 0$, which contradicts (12). Hence, there exists $y^* \in \mathcal{A} \backslash \partial\mathcal{A}$, such that $L^i(y^*) = F^i(y_i^*)$, $i = 1, 2$. $\square$

$(D_{10}, D_{11}, D_{00}, D_{01}) = (8, 5, 3, 1)$	$I = 100$
$\mu_\Lambda = 0.04$	$\sigma_\Lambda = (0.05, 0.05)$
$\mu_Y = (0.03, 0.03)$	$\Sigma_Y = \begin{bmatrix} 0.1 & 0 \\ 0.1 & 0.1 \end{bmatrix}$

Table 1: Model characteristics.

4.3 A Numerical Illustration

Consider the case with payoffs, sunk costs, and parameters as given in Table 1. It is assumed that both players have the same discount factor, $\Lambda^1 = \Lambda^2 \equiv \Lambda$. It is, furthermore, assumed that Player 1 is only influenced by z_1, whereas Player 2's payoffs are influenced by both shocks. This could correspond to a situation where Player 1 is a domestic firm, with an option to invest in a new product, and Player 2 is a foreign firm with a similar option. The Wiener process z_2 can represent, for example, exchange rate risk.

Figure 4 shows the regions S^i_N, S^i_P, and S^i_L for both players as well as a simulated sample path in which Player 2 exercises first at her optimal time. The path was simulated with $Y_0 = (0.1, 0.1) \in S^1_N \cap S^2_N$.

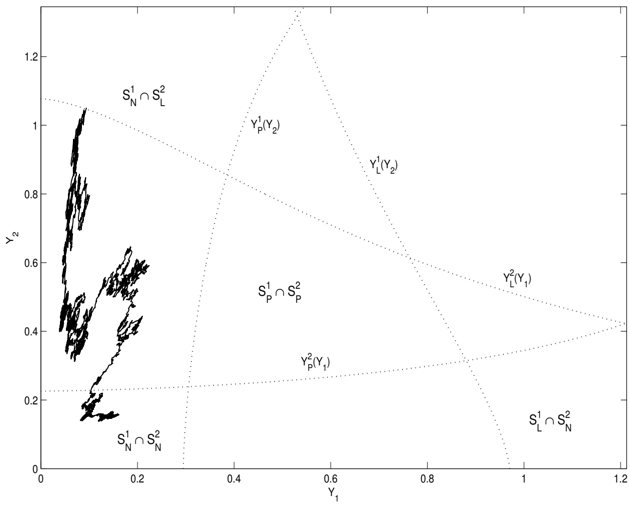

Figure 4: A sample path leading to sequential investment with Player 2 as leader.

Note that, since Y has continuous sample paths, in equilibrium there is always one player who does not exercise the option at time $\tau_{Y_0}(\Theta)$, a.s. It is, however, not the case that in the preemption region both players both exercise with probability 0.5, as is the case in papers where it is assumed that players toss a fair coin to determine who exercises first in the preemption region.[8] In fact, conditional on $\tau_{Y_0}(\Theta) = \tau_{Y_0}(\bar{S}_P^1 \cap \bar{S}_P^2)$, the probability that both players exercise with probability 0.5 is equal to 0. This is the case, because $P(X_\tau = (1,0)) = P(X_\tau = (0,1))$ only if $\varphi^1(Y_{\tau_{Y_0}(\Theta)}) = \varphi^2(Y_{\tau_{Y_0}(\Theta)}) = 0$, given that there is always one player for whom $\varphi^i(Y_{\tau_{Y_0}(\Theta)}) = 0$. There is only one point where this happens, namely at the intersection of $Y_P^1(Y_2)$ and $Y_P^2(Y_1)$. Due to absolute continuity, this point is reached with probability 0.

5 Conclusion

In this paper I have introduced a general model for two-player non-exclusive real option games. The strategy and equilibrium concepts are generalisations of Dutta and Rustichini (1995) and Fudenberg and Tirole (1985). The advantage of using the coordination of Fudenberg and Tirole (1985) is that it allows one to endogenously solve for a coordination problem, which often arises in preemption games. The basic idea is that if a coordination problem arises, the two players engage in a game in "artificial time", which leads to an absorbing Markov chain. The probabilities with which each player exercises the option is then simply given by the limit distribution of this chain. The main result of the paper, Theorem 1, proves the validity of the rent-equalisation principle in NEROs with first-mover advantages, where uncertainty is governed by a strong Markovian stochastic process.

Most of the present literature on game-theoretic real option models assumes that uncertainty is represented by a one-dimensional geometric Brownian motion. This paper shows that the results change significantly if a two-dimensional GBM is used. In much of the literature the coordination device is not used, but exogenous assumptions are made on the resolution of the coordination problem. Usually it is argued that a fair coin is tossed and each player exercises with probability 1/2. I suspect that such assumptions are based on Fudenberg and Tirole (1985) who show that this is the case in the particular (deterministic, symmetric players) model they study. In a purely symmetric model this is indeed true. With a 2-dimensional GBM, however, the coordination problem arises as well and, in equilibrium, neither player exercises with probability 1/2, almost surely. Furthermore, both players exercise with unequal probability, almost surely. It is still the case, however, that both players do not exercise simultaneously, almost surely, as is a standard assumption in the literature. This is due to the continuous sample paths of GBM.

The analysis in this paper opens up several avenues for future research. Firstly,

[8] See, for example, Grenadier (1996) or Weeds (2002)

the actual behaviour of the model for particular stochastic processes can be examined. Of particular interest would be the situation where Y follows a jump-diffusion process. In the models currently studied in the literature the probability of both players jointly exercising is zero, due to continuity of sample paths. This property would be lost in jump-diffusion model. This might consequently lead to an additional value of waiting.

Secondly, the model in Section 4 could be used to analyse specific economic problems. A straightforward one is the question whether currency unions, or currency pegging, accelerates investment. In the setting of Section 4 one can think of two firms, a domestic one (Player 1) and a foreign one (Player 2). The domestic firm is exposed to one source of risk, say product-market risk due to demand fluctuations, whereas the foreign firm is also exposed to exchange rate risk. A monetary union would take away the latter source of risk and lead to a duopoly as analysed in Huisman (2001, Chapter 7). The expected first and second exercise times could be simulated and a welfare analysis could be made to compare both situations.

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