Risk, Strategy, and Optimal Timing of M&A Activity*

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Abstract

In this paper, the problem of mergers and acquisitions under profit uncertainty is considered. A two firm model is developed where M&A activity is modelled as an act of risk diversification. We study the case where only the larger firm engages in M&A activity and the case where both firms do. It is shown that takeovers can be optimal during both economic expansions and contractions. The option value of M&A activity is determined. We argue that there is a minimum level of positive synergies for M&A activity to be optimal, which is increasing in the level of diversification. Furthermore, it is shown that under M&A competition, this option value vanishes completely and that hostile takeovers are never optimal. An analysis of optimal portfolio selection by a risk averse investor shows ambiguous wealth results of M&A activity.

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1 Introduction

One of the more influential investment decisions of a firm concerns the decision to acquire another firm. Influential since it not only changes the way a firm is structured and conducts its business, but also influences market structure and competition in an industry or product chain. Most of the theoretical literature on explaining M&A activity is concerned with (static) effects of competition and welfare changes due to horizontal mergers in Cournot or Bertrand oligopoly (cf. Salant et al. (1983) or Farrell and Shapiro (1990)). Recent papers on vertical mergers take a similar point of departure (cf. Martin and Schrader (1998)).

An important aspect of M&A activity is that is often takes place in waves during periods of economic expansions. Recent evidence on the merger waves is given in, for example, Maksimovic and Philips (2001). A theoretical explanation of procyclical merger waves is given in Lambrecht (2004). He uses a one-factor real options framework. That is, he considers an industry with two firms that face *identical* risk. In his view, a merger is a cooperative decision where the shares of both firms is determined according to Pareto optimality. So, firms decide on an optimal time first and then share the profits accordingly. In a (hostile) takeover the target first chooses its profit share upon which the acquirer determines the optimal time of takeover. Lambrecht (2004) concludes that M&A activity only takes place during periods of economic expansion.

The real options approach views the possibility of mergers and acquisitions as an option comparable to an American call option. The underlying asset here is a firm's discounted future profit stream. This profit stream is assumed to be subject to risk. In this paper we consider a two-factor model[1] with two expected profit maximising firms, which face different, but correlated, risk. This generalisation changes the qualitative nature of the analysis considerably compared to Lambrecht (2004). Firstly, his model fits horizontal mergers only, whereas our analysis can also capture vertical mergers. Secondly, procyclicality of merger waves is lost when one considers two factors.

The paper analyses two scenarios related to M&A activity. In the first scenario it is assumed that only one firm can engage in M&A activity. That is, a merger is always the result of one firm taking over the other firm. A takeover takes place as soon as one firm makes a bid on the other firm which the other firm does not reject. It is shown that the option value of M&A activity is positive. We find that

[1] In a recent paper Morellec and Zhdanov (2005) use a closely related two-factor model together with imperfect information to explain the empirically observed rise in asset prices around merger announcements.

M&A activity can take place both during economic expansions and contractions. The most important factor in determining the optimality of a merger or takeover decision is the *relative* profit of a firm vis à vis the other firm.

In the second scenario we consider the case where both firms can engage in M&A activity. A takeover, again, takes place if one firm makes an offer that is accepted by the other firm. A merger takes place if both firms make a bid simultaneously. In this case the profit shares are determined by a Nash bargaining procedure. We show that it is optimal for one firm to make a bid if and only if it is optimal for the other firm to make a bid as well. This result holds regardless of the relative size of the firms. Consequently, in equilibrium both firms will always simultaneously make a bid, i.e. only mergers occur in this scenario. Hence, (hostile) takeovers will never take place in equilibrium. Furthermore, the option value of M&A activity vanishes completely in case both firms can engage in it.

The analysis on optimal and strategic timing shows that the trade-off between synergies and risk at the *firm level* can lead to M&A activity both in periods of expansions and contractions. Is this optimal from the shareholder's point of view? A numerical study of optimal dynamic portfolio choice by a risk averse investors highlights a few issues. Firstly, even with high synergies, negative correlation does not lead to mergers being preferred by investors. Secondly, with positive correlation, on the other hand, even modest synergies make mergers the preferred option. Hence, the investor has a strong trade-off between synergies (higher expected value) and the reduction in diversification possibilities that both come with mergers. Thirdly, the investor's attitude to competition in M&A activity is ambiguous. Sometimes the investor prefers M&A competition (and, hence, an early merger) and sometimes no competition (and, hence, a late merger).

The paper is organised as follows. In Section 2 the case where only one firm can engage in M&A activity is analysed. In Section 3 we analyse the case where both firms can engage in M&A activity, whereas Section 4 discusses the desirability of M&A activity for shareholders.

2 The Optimal Timing of Acquisitions

Consider two firms, indexed by $i \in \{1, 2\}$, that operate in separate, but related markets. The profit flow of firm i at time $t \in [0, \infty)$, π_t^i, consists of a deterministic part, denoted by $D_i > 0$, and a stochastic component, denoted by $X_{i,t}$. The deterministic component can be thought of as resulting from competition in the product market.

The stochastic shock is assumed to be multiplicative, that is,

$$\pi_{i,t} = X_{i,t} D_i.$$

The stochastic shock follows a geometric Brownian motion with trend μ_i and volatility σ_i, i.e.

$$dX_{i,t} = \mu_i X_{i,t} dt + \sigma_i X_{i,t} dW_{i,t}, \tag{1}$$

where W_i is a Wiener process, so $dW_{i,t} \sim \mathcal{N}(0, dt)$. The instantaneous correlation between W_1 and W_2 equals $\rho \in (-1, 1)$. This implies that $dW_{i,t} dW_{i,t} = \rho dt$. It is assumed that the discount rate for both firms is equal to $r > 0$. Furthermore, in order for the problem to have a finite solution it is assumed that $\mu_i < r$, for $i \in \{1, 2\}$.

Suppose that firm 1 is the larger firm, which has an option to take over firm 2, leading to a combined deterministic profit flow $D_m > 0$. For simplicity, it is assumed that the takeover process does not involve sunk costs.[2] After the takeover it is assumed that the weight of market 1 in the new firm equals $\gamma \in (0, 1)$. Since firm 1 is the larger firm it is logical to assume that $\gamma \geq \frac{1}{2}$. So, the stochastic shock that the merged firm faces at time t, denoted by Y_t, equals[3]

$$Y_t = X_1^{\gamma} X_2^{1-\gamma}.$$

For further reference, the process $(Z_t)_{t \geq 0}$ is defined, where, for all $t \geq 0$, $Z_t = \frac{X_{1,t}}{X_{2,t}}$.

The following lemma states that Y and Z follow geometric Brownian motions. Its proof can be found in Appendix A.

Lemma 1 *There exists a Wiener process $(W_t^Y)_{t \geq 0}$, such that the stochastic process $(Y_t)_{t \geq 0}$ follows a geometric Brownian motion, equal to*

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dW_t^Y, \tag{2}$$

where

$$\mu_Y = \gamma \mu_1 + (1 - \gamma) \mu_2 - \frac{1}{2} \gamma (1 - \gamma) \left((\sigma_1 - \sigma_2)^2 + 2 \sigma_1 \sigma_2 (1 - \rho) \right), \tag{3}$$

$$\sigma_Y^2 = (\gamma \sigma_1 + (1 - \gamma) \sigma_2)^2 - 2 \gamma (1 - \gamma) \sigma_1 \sigma_2 (1 - \rho). \tag{4}$$

[2]Sunk costs of takeovers can be thought of to comprise, for example, the legal costs of the takeover (including the costs incurred for getting formal approval by competition authorities), the costs of restructuring the two organisations to facilitate the takeover, etc.

[3]This functional form is best understood by considering the deterministic case, i.e. $\sigma_1 = \sigma_2 = 0$. Then it holds that $X_t^i = e^{\mu_i t}$ for $i = 1, 2$. Hence, the growth rate of the profit of firm i equals μ_i. The growth rate of the merged firm should then equal $\gamma \mu_1 + (1 - \gamma) \mu_2$. In other words, $Y = e^{\gamma \mu_1 + (1-\gamma) \mu_2} = X_1^{\gamma} X_2^{1-\gamma}$.

Furthermore, there exists a Wiener process $(W_t^Z)_{t\geq 0}$*, such that the stochastic process* $(Z_t)_{t\geq 0}$ *follows a geometric Brownian motion, equal to*

$$dZ_t = \mu_Z Z_t dt + \sigma_Z Z_t dW_t^Z, \tag{5}$$

where

$$\mu_Z = \mu_1 - \mu_2 + \sigma_2(\sigma_2 - \sigma_1\rho), \tag{6}$$

$$\sigma_Z^2 = (\sigma_1 + \sigma_2)^2 - 2\sigma_1\sigma_2(1+\rho). \tag{7}$$

Note that $\mu_Y < r$. Furthermore, it holds that $\mu_Y < \gamma\mu_1 + (1-\gamma)\mu_2$. Hence, the trend of the uncertainty faced by the merged firm is lower than the weighted average of the trends of the separate firms. This is offset, though, by a smaller volatility, since $\sigma_Y^2 < (\gamma\sigma_1 + (1-\gamma)\sigma_2)^2$. Hence, a takeover can be seen as an act of diversification, comparable to an investor creating a portfolio with different assets to diversify risk. An important question is whether firms should engage in M&A activity purely for diversifying risk. In standard Corporate Finance texts it is argued they should not (see e.g. Brealey and Myers (2003)). We return to this important issue in Section 4.

It is assumed throughout that each firm maximises expected discounted profits. In complete and efficient markets this represents the market value of the firm. In contrast to Morellec and Zhdanov (2005) it is assumed that shareholders have perfect information. If the acquirer decides to takeover the target at time t, the value to its shareholders is denoted by $V(X_{1t}, X_{2t})$.

Suppose that firm 1 decides to take over firm 2 at time $\tau \geq 0$. Then firm 1 has to compensate the shareholders of firm 2 for "'losing" their firm. The profit stream of the newly formed firm will be YD_m, while the stand-alone profit stream of firm 2 equals $X_2 D_2$. So, the management of firm 1 should offer the shareholders of firm 2 a profit share $s_\tau \in [0,1]$, such that the expected discounted value of the new firm is at least as high as the expected discounted stand-alone value. That is, s_τ should be such that

$$\mathbb{E}\Big(\int_\tau^\infty e^{-rt} s_\tau Y_t D_m dt\Big) \geq \mathbb{E}\Big(\int_\tau^\infty e^{-rt} X_{2t} D_2 dt\Big). \tag{8}$$

Since the management of firm 1 maximises its own market value, (8) holds with equality in an optimum. Standard computations[4] show that $\mathbb{E}(\int_t^\infty e^{-rs} Y_s ds) = \frac{Y_t}{r-\mu_Y}$. Hence, solving (8) gives

$$\begin{aligned} s_\tau &= \frac{D_2}{D_m}\frac{r-\mu_Y}{r-\mu_2}\frac{X_{2\tau}}{Y_\tau} \\ &= \frac{D_2}{D_m}\frac{r-\mu_Y}{r-\mu_2}\Big(\frac{X_{2\tau}}{X_{1\tau}}\Big)^\gamma. \end{aligned} \tag{9}$$

[4]See e.g. Dixit and Pindyck (1994).

The expected discounted value of the acquisition at time $t \geq 0$, is, therefore, equal to

$$\begin{aligned} V(X_{1t}, X_{2t}) &= \mathbb{E}\Big(\int_{\tau}^{\infty} e^{-rt}(1-s_t)Y_t D_m dt\Big) \\ &= \frac{D_m}{r-\mu_Y} Y_t - \frac{D_2}{r-\mu_2}\Big(\frac{X_{1t}}{X_{2t}}\Big)^{-\gamma} Y_t \\ &= X_{2t}\Big[\frac{D_m}{r-\mu_Y} Z_t^{\gamma} - \frac{D_2}{r-\mu_2}\Big]. \end{aligned} \tag{10}$$

Let $\mathcal{T}$ denote the set of stopping times for $(X_t)_{t\geq 0}$, where $X_t = (X_{1t}, X_{2t})$, for all $t \geq 0$. The problem for firm 1 is to solve the following optimal stopping problem: Find $G^*(x_1, x_2)$ and $T^* \in \mathcal{T}$ such that

$$\begin{aligned} G^*(x_1, x_2) &= \sup_{T\in\mathcal{T}} \mathbb{E}\Big[\int_0^T e^{-rt} D_1 X_{1t} dt + e^{-rT} V(X_{1T}, X_{2T})\Big] \\ &= \mathbb{E}\Big[\int_0^{T^*} e^{-rt} D_1 X_{1t} dt + e^{-rT^*} V(X_{1T^*}, X_{2T^*})\Big]. \end{aligned} \tag{11}$$

Proposition 1 *Let β_1 and β_2 be the positive and negative root, respectively, of the quadratic equation*

$$\mathcal{Q}(\beta) \equiv \frac{1}{2}\sigma_Z^2 \beta(\beta-1) + (\mu_1 - \mu_2)\beta - (r-\mu_2) = 0.$$

Furthermore, suppose that

$$\gamma \frac{D_m}{r-\mu_Y} > \Big(\frac{D_1}{r-\mu_1}\Big)^{\gamma} \Big(\frac{\gamma}{1-\gamma}\frac{D_2}{r-\mu_2}\Big)^{1-\gamma}. \tag{12}$$

It holds that there exist pairs $(\underline{A}, \underline{Z})$ and $(\overline{A}, \overline{Z})$ such that the optimal stopping problem (11) is solved by $(G^*(\cdot), T^*)$, where

$$G^*(x_1, x_2) = \begin{cases} x_2\Big(\underline{A}\Big(\frac{x_1}{x_2}\Big)^{\beta_1} + \frac{D_1}{r-\mu_1}\frac{x_1}{x_2}\Big) & \text{if } 0 \leq \frac{x_1}{x_2} < \underline{Z} \\ \frac{D_m}{r-\mu_Y} x_1^{\gamma} x_2^{1-\gamma} - \frac{D_2}{r-\mu_2} x_2 & \text{if } \underline{Z} \leq \frac{x_1}{x_2} \leq \overline{Z} \\ x_2\Big(\overline{A}\Big(\frac{x_1}{x_2}\Big)^{\beta_2} + \frac{D_1}{r-\mu_1}\frac{x_1}{x_2}\Big) & \text{if } \frac{x_1}{x_2} > \overline{Z}, \end{cases} \tag{13}$$

and

$$T^* = \inf\{t \geq 0 | Z_t \in [\underline{Z}, \overline{Z}]\}. \tag{14}$$

Proof. Instead of the standard method of solving the optimal stopping problem (11) via the Bellman equation (cf. Dixit and Pindyck (1994)) we use the fact that (11) is similar to the Dirichlet problem with free boundary. For details on the background, see Appendix B or Øksendal (2000, Chapter 10).

The problem (11) is not time-homogeneous. Consider, therefore, the stochastic process $B_t = (s+t, X_{1t}, X_{2t}, P_t)$, defined by

$$dB_t = \begin{bmatrix} 1 \\ \mu_1 X_{1t} \\ \mu_2 X_{2t} \\ e^{-rt} D_1 X_{1t} \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sigma_1 X_{1t} \\ \sigma_2 X_{2t} \\ 0 \end{bmatrix} dW_t,$$

where W_t is a 4-dimensional Brownian motion. Then

$$G^*(x_1, x_2) = \sup_{T \in \mathcal{T}} \mathbb{E}[P_T + e^{-rT} V(X_{1T}, X_{2T})] = \sup_{T \in \mathcal{T}} \mathbb{E}[G(B_T)], \tag{15}$$

with

$$G(b) = e^{-rs} V(x_1, x_2) + p.$$

The optimal stopping problem (15) is a time-homogeneous optimal stopping problem that is equivalent to (11). Therefore, we can apply Øksendal (2000, Theorem 10.4.1) (see Appendix B) to problem (15).

Pivotal in the proof is the following lemma, the proof of which can be found in Appendix C.

Lemma 2 *If the inequality (12) holds, then the following systems of equations permit a solution in* (A, Z)*,*

$$A_1 Z_1^{\beta_1} + \frac{D_1}{r - \mu_1} Z_1 = \frac{D_m}{r - \mu_Y} Z_1^{\gamma} - \frac{D_2}{r - \mu_2} \tag{16}$$

$$A_1 \beta_1 Z_1^{\beta_1 - 1} + \frac{D_1}{r - \mu_1} = \gamma \frac{D_m}{r - \mu_Y} Z_1^{\gamma - 1}, \tag{17}$$

and

$$A_2 Z_2^{\beta_2} + \frac{D_1}{r - \mu_1} Z_2 = \frac{D_m}{r - \mu_Y} Z_2^{\gamma} - \frac{D_2}{r - \mu_2} \tag{18}$$

$$A_2 \beta_2 Z_2^{\beta_2 - 1} + \frac{D_1}{r - \mu_1} = \gamma \frac{D_m}{r - \mu_Y} Z_2^{\gamma - 1}, \tag{19}$$

where the solutions are such that $A_1 > 0$ *and* $A_2 > 0$*. Furthermore, there exists* $\bar{Z}$ *such that* $Z_1 < \bar{Z} < Z_2$*.*

We state that the continuation region is of the form $D = \{(s, x_1, x_2) | 0 < \frac{x_1}{x_2} < Z_1\} \cup \{(s, x_1, x_2) | \frac{x_1}{x_2} > Z_2\}$, for some $Z_1 > 0$ and $Z_2 > 0$, such that $Z_1 < \bar{Z} < Z_2$. Define $\tau_D := \inf\{t \geq 0 | B_t \notin D\}$. We compute

$$F(s, x_1, x_2, p) = \mathbb{E}[G(\tau_D)].$$

From Øksendal (2000, Theorem 9.2.14) we know that F solves the Dirichlet problem, i.e. it is the bounded solution to the boundary value problem

$$\begin{cases} \mathcal{L}_X F = 0 \quad \text{in } D \\ \lim\limits_{x_1/x_2 \to Z^*} F(s, x_1, x_2) = g(s, Z^*) \end{cases},$$

where $\mathcal{L}_X$ is the partial differential operator. That is, we have

$$\begin{aligned} \mathcal{L}_{(X_1,X_2)}F =& \frac{\partial F}{\partial s} + \mu_1 X_1 \frac{\partial F}{\partial x_1} + \mu_2 X_2 \frac{\partial F}{\partial x_2} + e^{-rs} x_1 D_1 \frac{\partial F}{\partial p} \\ &+ \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 F}{\partial x_1^2} + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 F}{\partial x_2^2} + \sigma_1 \sigma_2 \rho x_1 x_2 \frac{\partial^2 F}{\partial x_1 \partial x_2} = 0. \end{aligned} \tag{20}$$

If we impose that $F(\cdot)$ is of the form

$$F(s, x_1, x_2, p) = e^{-rs} x_2 \varphi(z) + p,$$

with $z = x_1/x_2$, the partial derivatives of $F(\cdot)$ become $\frac{\partial F}{\partial s} = -re^{-rs}x_2\varphi(z)$, $\frac{\partial F}{\partial x_1} = e^{-rs}\varphi'(z)$, $\frac{\partial F}{\partial x_2} = e^{-rs}(\varphi(z) - z\varphi'(z))$, $\frac{\partial^2 F}{\partial x_1^2} = e^{-rs}\varphi''(z)/x_2$, $\frac{\partial^2 F}{\partial x_2^2} = e^{-rs}z^2\varphi''(z)/x_2$, $\frac{\partial^2 F}{\partial x_1 \partial x_2} = -e^{-rs}z\varphi''(z)/x_2$, and $\frac{\partial F}{\partial p} = 1$. Hence, (20) becomes

$$\begin{aligned} \mathcal{L}_{(X_1,X_2)}F = e^{-rs}x_2\Big[&- r\varphi(z) + \mu_1 z\varphi'(z) + \mu_2\big(\varphi(z) + z\varphi'(z)\big) + zD_1 \\ &+ \frac{1}{2}\sigma_1^2 z^2 \varphi''(z) + \frac{1}{2}\sigma_2^2 z^2 \varphi''(z) - \sigma_1\sigma_2\rho z^2 \varphi''(z)\Big] = 0 \\ \iff \frac{1}{2}\sigma_Z^2 z^2 \varphi''(z) + (\mu_1 - \mu_2)z\varphi'(z) &- (r - \mu_2)\varphi(z) + zD_1 = 0. \end{aligned} \tag{21}$$

The partial differential equation (21) has the general solution

$$\varphi(z) = A_1 z^{\beta_1} + A_2 z^{\beta_2} + \frac{D_1}{r - \mu_1} z,$$

where β_1 and β_2 solve $\mathcal{Q}(\beta) = 0$, and A_1 and A_2 are constants. The boundedness condition on the solution implies that it should hold that $\lim_{z \downarrow 0} \varphi(z) = 0$ and $\lim_{z \to \infty} \varphi(z) = 0$.[5]. Since $\beta_1 > 1$ and $\beta_2 < 0$, it should, therefore, hold that $A_2 = 0$ on $[0, \bar{Z})$ and $A_1 = 0$ on $(\bar{Z}, \infty)$.

If (A_1, Z_1) and (A_2, Z_2) satisfy the boundary conditions (16) and (18), respectively, a candidate solution for (11) is obtained:

$$F(t, x_1, x_2) = \begin{cases} e^{-rt}\left(x_2 A_1 \left(\frac{x_1}{x_2}\right)^{\beta_1} + \frac{D_1}{r-\mu_1} x_1 \right) & \text{if } 0 < \frac{x_1}{x_2} < Z_1 \\ e^{-rt}\left(\frac{D_m}{r-\mu_Y} x_1^{\gamma} x_2^{1-\gamma} - \frac{D_2}{r-\mu_2} x_2 \right) & \text{if } Z_1 \le \frac{x_1}{x_2} \le Z_2 \\ e^{-rt}\left(x_2 A_2 \left(\frac{x_1}{x_2}\right)^{\beta_2} + \frac{D_1}{r-\mu_1} x_1 \right) & \text{if } \frac{x_1}{x_2} > Z_2. \end{cases}$$

[5]These conditions also rule out the existence of speculative bubbles. See Dixit and Pindyck (1994, Section 6.1.C)

If (A_1, Z_1) and (A_2, Z_2) in addition satisfy the smooth pasting conditions (17) and (19), respectively, it holds that $\varphi \in C^1$.

It is easy to see that B_t spends 0 time on ∂D a.s., that ∂D is a Lipschitz surface, that $\varphi \in C^2(\mathbb{R}\backslash\partial D)$ with locally bounded second order derivatives near ∂D, that $\tau_D < \infty$ a.s., and that the family $\{\varphi(Y_\tau)|\tau < \tau_D\}$ is uniformly integrable for all $y \in \mathbb{R}$. By construction, it holds that $\mathcal{L}_X\varphi = 0$ on D, where $\mathcal{L}_X\varphi$ is the partial differential operator of φ (see Appendix B). Furthermore, from

$$\begin{aligned}
\mathcal{L}_{(X_1,X_2)}F =& e^{-rs}\frac{D_m}{r-\mu_Y}x_1^\gamma x_2^{1-\gamma}\Big(-r + \gamma\mu_1 + (1-\gamma)\mu_2 \\
& - \frac{1}{2}\gamma(1-\gamma)(\sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho)\Big) \\
=& - e^{-rs}x_2\frac{D_2}{r-\mu_2}(r+\mu_2) \\
<& - e^{-rs}\frac{D_m}{r-\mu_Y}x_1^\gamma x_2^{1-\gamma}(r-\mu_Y) \\
<& 0,
\end{aligned}$$

it follows that $\mathcal{L}_X\varphi \leq 0$, for $\frac{x_1}{x_2} \notin D$. Finally, we can see that $\varphi(\cdot) \geq V(\cdot)$, which follows immediately from the following lemma, which is a direct corollary to Lemma 2.

Lemma 3 *Define* $f_1(z) = A_1 z^{\beta_1} + \frac{D_1}{r-\mu_1}z$*,* $f_2(z) = A_2 z^{\beta_2} + \frac{D_1}{r-\mu_1}z$*, and* $g(z) = \frac{D_m}{r-\mu_Y}z^\gamma - \frac{D_2}{r-\mu_2}$*. It holds that, if* $0 \leq x_1/x_2 < Z_1$*, then* $f_1(z) > g(z)$*. If* $x_1/x_2 > Z_2$*, then* $f_2(z) > g(z)$*.*

Since all conditions of Theorem B.1 are satisfied, the pair $(F(\cdot), \tau_D)$ solves the optimal stopping problem (11). □

From Proposition 1 it becomes clear that not the *absolute* profitability of firms is important, but *relative* profitability. It, therefore, does not follow directly that takeovers take place during economic booms. This results from the fact that Z^* is reached either from below on $[0, \underline{Z})$, or from above on $(\overline{Z}, \infty)$. On $[0, \underline{Z})$ a takeover can take place either if firm 1 experiences a sharper upswing than firm 2, or a slower downturn. In both cases Z is increasing. On $(\overline{Z}, \infty)$ a takeover can take place if firm 1 experiences a sharper downturn that firm 2 or a slower upswing. In both cases Z is decreasing. Note that Lambrecht (2004) concludes unequivocally that mergers only take place during economic booms. This happens because in his model both firms are subject to the same random process from the outset.

An important result of this model is that takeovers can only be optimal if the synergies are high enough. To see this consider a case where $\mu_1 = \mu_2 \equiv \mu$, $\gamma = \frac{D_1}{D_1+D_2}$, and $D_m = (1+\alpha)(D_1+D_2)$ is a synergy parameter. These synergies can arise from increased production efficiency or a decrease in competition, or a combination of both. It is easy to see that, in this case, (12) holds iff $\alpha > \underline{\alpha} \equiv \frac{r-\mu_Y}{r-\mu} - 1$.[6] Note that the lower bound $\underline{\alpha} \geq 0$ for all feasible parameter configurations. Furthermore, $\underline{\alpha}$ is decreasing in ρ, with $\underline{\alpha} = 0$ for $\rho = 1$. In other words, the higher the degree of diversification (i.e. the smaller ρ) the higher the minimally required synergies. The intuition behind this result is that the firm is, in fact, risk neutral. That is, it does not care about volatility. The higher ρ, the lower the volatility σ_Y and the lower the trend μ_Y. In order to offset the reduction in trend and, hence, expected discounted profits, the higher the synergies need to be. That is, the diversification argument is not important for a risk neutral firm.

3 The Strategic Timing of Mergers and Acquisitions

In this section, the model from the previous section is extended to a situation where both firms can decide to make an acquisition offer. Throughout, it is assumed that if both firms simultaneously make an offer, a merger is agreed upon. We follow the basic setup for simple timing games as described in Fudenberg and Tirole (1991, Section 4.5).

Each firm has the choice to make an acquisition offer at each point in time t. So, the strategy set for firm i at time t is

$$A_i(t) = \{\text{make offer}, \text{don't make offer}\}.$$

Suppose that at time t, firm 1 makes an acquisition offer to firm 2. In the terminology of timing games this makes firm 1 the "leader". Firm 2 is the "follower" in this case. The payoff to firms 1 and 2 are (cf. (10))

$$L_1(X_{1t}, X_{2t}) = X_{2t}\Big[\frac{D_m}{r-\mu_Y} Z_\tau^\gamma - \frac{D_2}{r-\mu_2}\Big]$$

and

$$F_2(X_{1t}, X_{2t}) = X_{1t}\Big[\frac{D_2}{r-\mu_2}\frac{1}{Z_t}\Big],$$

respectively. In the case firm 2 makes an acquisition offer, while firm 1 does not, the payoffs are given by

$$L_2(X_{1t}, X_{2t}) = X_{1t}\Big[\frac{D_m}{r-\mu_Y}\Big(\frac{1}{Z_t}\Big)^{1-\gamma} - \frac{D_1}{r-\mu_1}\Big]$$

and

$$F_1(X_{1t}, X_{2t}) = X_{2t}\Big[\frac{D_1}{r-\mu_1} Z_t\Big],$$

[6] It holds that if $\alpha = \underline{\alpha}$, then $\underline{Z} = \bar{Z} = 1$.

respectively.

If both firms simultaneously make an acquisition offer at time t it is assumed that a merger takes place. The firms use the Nash bargaining solution (Nash (1950)) with disagreement point $d = \left(\frac{D_1}{r-\mu_1}X_{1t}, \frac{D_2}{r-\mu_2}X_{2t}\right)$ to determine how to split the value $\frac{D_m}{r-\mu_Y}Y_t$. The bargaining power of firm 1 is assumed equal to its relative market power, γ. It is easily shown that this leads to the merger payoffs

$$\begin{aligned} M_1(X_{1t}, X_{2t}) &= \frac{D_m}{r-\mu_Y}Y_t + \frac{1}{2}\Big(\frac{D_1}{r-\mu_1}X_{1t} - \frac{D_2}{r-\mu_2}X_{2t}\Big) \\ &= \gamma L_1(X_{1t}, X_{2t}) + (1-\gamma)F_1(X_{1t}, X_{2t}), \end{aligned}$$

and

$$\begin{aligned} M_2(X_{1t}, X_{2t}) &= \frac{D_m}{r-\mu_Y}Y_t + \frac{1}{2}\Big(\frac{D_2}{r-\mu_2}X_{2t} - \frac{D_1}{r-\mu_1}X_{1t}\Big) \\ &= (1-\gamma)L_2(X_{1t}, X_{2t}) + \gamma F_2(X_{1t}, X_{2t}), \end{aligned}$$

respectively.

The following lemma determines the region where the leader payoff is larger, respectively smaller, for both firms. The proof can be found in Appendix D.

Lemma 4 *Suppose that (12) holds. Then there exists an interval $D_P = [Z_1, Z_2]$, for certain Z_1 and Z_2, such that*

$$Z \in D_P \iff L_1(Z) \geq F_1(Z) \text{ and } L_2(Z) \geq F_2(Z).$$

Lemma 4 shows that there exists values of Z where both firms want to be the leader. Even stronger: Firm 1 wants to acquire firm 2 if and only if firm 2 wants to acquire firm 1. This result holds *irrespective* of the relative market power parameter γ.

In the region D_P it holds that $L_i(x_1, x_2) \geq M_i(x_1, x_2) \geq F_i(x_1, x_2)$, with strict inequalities in the interior. At each point in time both firms basically play the state game depicted in Figure 1. If $z \notin D_P$, not making an offer is a dominant strategy for

	make offer	don't make offer
make offer	$\big(M_1(X_{1t}, X_{2t}), M_2(X_{1t}, X_{2t})\big)$	$\big(L_1(X_{1t}, X_{2t}), F_2(X_{1t}, X_{2t})\big)$
don't make offer	$\big(F_1(X_{1t}, X_{2t}), L_2(X_{1t}, X_{2t})\big)$	$\big(F_1(X_{1t}, X_{2t}), F_2(X_{1t}, X_{2t})\big)$

Figure 1: The state game.

both firms. For $z \in D_P$, making an offer is a (weakly) dominant strategy for both firms. Let $T_P = \inf\{t \geq 0 | Z_t \in D_p\}$. Note that $T_P(\omega) \in \bar{\mathbb{R}}$ for all $\omega \in \Omega$. A strategy for firm i consists of a distribution function $G_i : \mathbb{R}_+ \to [0, 1]$, where $G_i(t)$ is the probability that firm i has invested before time t. It is easily seen that a subgame perfect equilibrium (in weakly dominant strategies) is given by

$$G_i(t) = \begin{cases} 0 & \text{if } 0 \leq t < T_P \\ 1 & \text{if } t \geq T_P. \end{cases} \tag{22}$$

From (22) it follows that the option value, which in the one firm case is given by $\underline{A}Z^{\beta_1}$ or $\overline{A}Z^{\beta_2}$, completely disappears when both firms can acquire each other. This contrary to the standard real options literature where competition erodes the option value, albeit it does not vanish completely (cf. Thijssen (2004, Chapter 4)). In the case of M&A activity the option value completely vanishes, because this value can be thought of as resulting from a zero-sum game: One firm's gain is the other firm's loss. In order to acquire the other firm, a firm must pay the other firm's shareholders its expected discounted stand-alone value. Furthermore, it forgoes its own expected stand-alone value. The expected discounted value of the merged firm offsets this stand-alone value if and only if this holds for the other firm as well. Therefore, it is optimal fir firm 1 to acquire firm 2 if and only if it is optimal for firm 2 to acquire firm 1. That is, we should only observe friendly mergers in a market. A hostile takeover is (in this framework) always a dominated strategy.

The fact that (22) holds irrespective of the market power parameter γ is caused by the assumption that if both firms choose to make a bid in the game depicted in Figure 1, the division of the profits is given by the asymmetric Nash bargaining solution, where the bargaining power of firm 1 equals γ, so that the effect of γ is internalised. Furthermore, the disagreement point is not a credible option in the region D_P, since if firm i is acquired in region D_P it gets exactly its expected discounted stand-alone value, whereas the Nash bargaining solution always gives *at least* this value.

4 M&A Activity and Shareholders

An important question concerning M&A activity is whether it is good for the shareholders. Corporate Finance texts like Brealey and Myers (2003, Chapter 33) mention "good" and "wrong" reasons to pursue M&A activity. Among the good reasons is the quest for synergies, whereas one of the bad reasons is the search for diversification of risk.[7] Intuitively, it is not in the shareholder's interest if a firm diversifies

[7]Note that part of these synergies may be due to reduced competition and can, hence, be socially undesirable. In fact, many mergers result from management's desire to increase market power. In this light M&A activity plays virtually the same role as R&D in Schumpeter (1942). One could even argue that the standard general equilibrium paradigm of price taking firms is incompatible

risk, since the shareholder can do this herself by choosing an appropriate portfolio of stocks in the two firms. In addition, a merger or acquisition implies the firms factually making a fixed portfolio choice for all the shareholders. Hence, the shareholders loose the possibility of diversifying risk according to their needs. Therefore, M&A activity should yield enough synergies so as to offset this loss of flexibility.

In Section 2 it was argued that firms which maximise expected discounted profits and have an optimal M&A policy require a minimum level of synergies. Even in the case of strategic M&A activity this minimum level of synergies should be obtained. The reason is that an expected profit maximising firm does not care about volatility and, hence, diversification. This seems to indicate that, indeed, firms engage in M&A activity if this is optimal from the point of view of shareholders.

The picture is, however, more complicated. If markets are complete and investors are risk-neutral then the analysis in Sections 2 and 3 describes a policy that is optimal from the point of view of the shareholders. However, the assumption of risk-neutrality seems to be very unrealistic. One reason being that if investors are risk-neutral they would only hold assets in the firm with the highest expected discounted profit stream. That is, one of the firms would a.s. not have any shareholders. This is obviously not the case in real-life financial markets.

In case investors are risk-averse the story is more complicated. The analysis presented in this paper would yield the same results as for risk-neutral investors if markets are complete and if the M&A option is valued under the equivalent martingale measure. The difficulty here, of course, is how to determine this measure. Under complete markets the equivalent martingale measure is uniquely determined.[8] This basically means that all investors agree on (contingent) state prices. This occurs precisely because markets are complete and, hence, there exists a full set of Arrow securities so that all risk in the economy can be traded and, therefore, properly priced. If markets are incomplete, then there exist multiple equivalent martingale measures. This means that investors do not agree on the appropriate price of risk in certain states of the world and that their level of disagreement depends on their personal risk attitude. In other words, in order to properly value the M&A option a firm has to know the risk attitudes of all its shareholders. In the literature on financial options numerous procedures have been proposed to price derivatives in incomplete markets and how to choose among the plethora of equivalent martingale measures. For example, measures are chosen based on minimising the variance (cf. Schweizer (1992)), or minimising the entropy (cf. Fritelli (2000)). A more recent

with capitalism as is partly illustrated by the desire of many firms in many industries to engage in M&A activity.

[8]This follows from the Girsanov theorem. See, for example, Dana and Jeanblanc (2003).

approach recognises that risk can only be hedged to a certain extent (for example 95%) and uses this fact to price the option (cf. Fölmer and Leukert (1999)).

Another possibility to judge the desirability of M&A activity is to draw a parallel with mean-variance efficiency (MVE). The problem here is how to incorporate the dynamics of the model into an MVE analysis. Since risk is modelled here to evolve according to a geometric Brownian motion, the variance of the sample paths is increasing with time. This implies that, although the deterministic profit flow of a merged firm might be higher than the sum of the parts due to synergy effects, this higher deterministic part also increases the variance in a quadratic way. Making a good MVE analysis is, therefore, not an easy task.

One way to think about the desirability of M&A activity is by using optimal portfolio theory. The idea is to compare the optimal wealth of an investor who can invest in a riskless asset and in stocks of both firms with the optimal wealth of the same investor in case she can only invest in a riskless asset or stocks of the merged firm. We will follow Duffie (1996, Section 9.D) and assume an investor with preferences over consumption, c, exhibiting constant relative risk aversion (CRRA):

$$u(c) = \frac{c^{1-\nu}}{1-\nu}.$$

It is assumed that the investor's discount rate is $r = 0.1$. The interest rate on the riskless asset equals $\delta = 0.04$, $\mu = \mu_1 = \mu_2 = 0.06$, and $\sigma 1 = \sigma 2 = 0.2$. This means that the excess return on the two risky assets equals $\lambda = (0.02, 0.02)$. We take $D_1 = 100$, $D_2 = 50$, $\gamma = \frac{D_1}{D_1+D_2}$, and $D_m = (1+\alpha)(D_1 + D_2)$, where α is the synergy parameter.

The stochastic processes $(X_{1t})_{t\geq 0}$ and $(X_{2t})_{t\geq 0}$ (and, hence, Y and Z) are defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$. Let $c_t \in \mathbb{R}_+$ and $\varphi_t \in \mathbb{R}^2$ denote the consumption rate and the fractions of wealth invested in risky assets at time t, respectively. Given initial wealth w, let $C(w)$ denote the set of adapted processes $(c_t, \varphi_t)_{t\geq 0}$, with $\int_0^\infty c_t dt < \infty$ (a.s.) and $\int_0^\infty (\varphi'\varphi)_t dt < \infty$ (a.s.). From Duffie (1996, Section 9.D) it follows that the optimal wealth at time 0 of investing w equals

$$\begin{aligned} J(w) &= \sup_{(c_t,\varphi_t)_{t\geq 0}\in C(w)} \mathbb{E}_{(X_1,X_2)}\left(e^{-rt}\int_0^\infty \frac{c_t^{1-\nu}}{1-\nu}dt\right) \\ &= \left(\frac{r-\delta(1-\nu)}{\nu} - \frac{(1-\nu)\lambda'\Sigma^{-1}\lambda}{2\nu^2}\right)^{-\nu}\frac{w^{1-\nu}}{1-\nu}, \end{aligned}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{bmatrix}$$

Similarly, optimal wealth at time 0 of investing w in case of a merged firm equals

$$
\begin{aligned}
J_m(w) = & \sup_{(c_t,\varphi_t)_{t\geq 0}\in C(w)} \mathbb{E}_Y\Big(\int_0^\infty e^{-rt}\frac{c_t^{1-\nu}}{1-\nu}dt\Big) \\
= & \Big(\frac{r-\delta(1-\nu)}{\nu} - \frac{(1-\nu)(\lambda_m^2/\sigma_Y^2)}{2\nu^2}\Big)^{-\nu}\frac{w^{1-\nu}}{1-\nu},
\end{aligned}
$$

where $\lambda_m = \mu_Y - \delta$ is the excess return of the merged firm.

If a merger takes place at the stopping time T it is assumed that at time T the investor's initial wealth w is "upgraded" to $(1+\alpha)w$. Let c^* en c_m^* denote the optimal consumption stream without and with merger, respectively. Then the optimal wealth at time 0 of an investor equals

$$
\begin{aligned}
V(T,w) = & \mathbb{E}_{(X_1,X_2)}\Big(\int_0^T e^{-rt}\frac{(c_t^*)^{1-\nu}}{1-\nu}dt\Big) + \mathbb{E}_Y\Big(\int_T^\infty e^{-rt}\frac{(c_{mt}^*)^{1-\nu}}{1-\nu}dt\Big) \\
= & \mathbb{E}_{(X_1,X_2)}\Big(\int_0^\infty e^{-rt}\frac{(c_t^*)^{1-\nu}}{1-\nu}dt\Big) - \mathbb{E}_{(X_1,X_2)}\Big(\int_T^\infty e^{-rt}\frac{(c_t^*)^{1-\nu}}{1-\nu}dt\Big) \\
& + \mathbb{E}_Y\Big(\int_T^\infty e^{-rt}\frac{(c_{mt}^*)^{1-\nu}}{1-\nu}dt\Big) \\
= & J(w) + \mathbb{E}(e^{-rT})(J_m(w) - J(w)).
\end{aligned}
$$

Let T_Z and T^Z denote the stopping times $T_Z = \inf\{t \geq 0 | Z_t \geq Z\}$ and $T^Z = \inf\{t \geq 0 | Z_t \leq Z\}$. We numerically assess the optimal wealth at time 0 of investing \$1 as a function of ρ in three cases. In the first case we consider the scenario that no merger is allowed, i.e. $J(1)$. Suppose that $Z_0 < Z_1$. The second case is when only firm 1 engages in M&A activity, i.e. we compute $V(T_{\underline{Z}}, 1)$. In the third case we allow both firms to be active and compute $V(T_{Z_1}, 1)$. The analysis is repeated for the case when $Z_0 > Z_2$, i.e. $J(1)$, $V(T^{\bar{Z}}, 1)$, and $V(T^{Z_2}, 1)$ are computed. We let ρ run from -0.8 to 1. In all cases we take $\nu = 0.7$. As starting values we choose $Z_0 = 0.5Z_1$ and $Z_0 = 2Z_2$, respectively. As synergy value we take $\alpha = \frac{r-\mu_Y}{r-\mu} - 0.95$. This produces Figure 2. The following lemma has been used to obtain this figure.

Lemma 5 *It holds that:*
1. If $z < Z^*$, then $\mathbb{E}\big[e^{-rT_{Z^*}}\big] = \big(\frac{z}{Z^*}\big)^{\beta_1}$.
2. If $z > Z^*$, then $\mathbb{E}\big[e^{-rT^{(Z^*)}}\big] = \big(\frac{z}{Z^*}\big)^{\beta_2}$.

The proof of this Lemma can be found in Appendix E.

From Figure 2 one can see that for low values of ρ all three cases yield a similar wealth. This is striking if one realises that in these cases the synergy value that is used is particularly high.[9] For higher values of ρ, $J(1)$ decreases because the

[9] For $\rho = -0.8$ the synergy from a merger is assumed to be 65% in order to ensure existence of Z_1 and z_2.

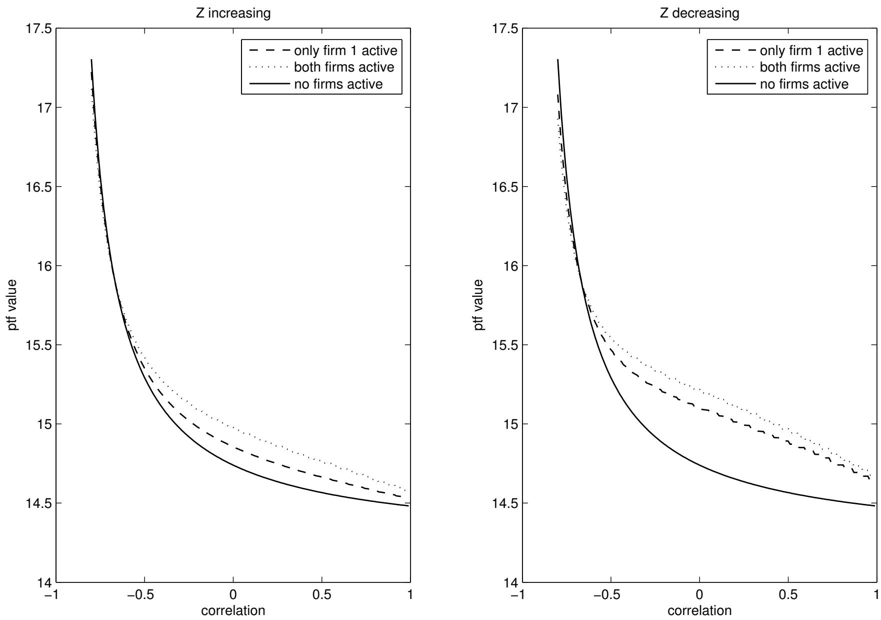

Figure 2: Optimal portfolio value as function of ρ.

there are less diversification possibilities. The other lines are decreasing because the synergy value α decreases. One notes that in both graphs there is an increasing gap between $J(1)$ and the other lines which becomes smaller for large values of ρ. This happens because for high values of ρ, α decreases to 5%. The processes X_1, X_2, and Y become indistinguishable. The gap is, therefore, entirely due to the 5% synergy. The parabolic nature of the gap between $J(1)$ and the other lines suggests there is an maximum point where the merger synergy outweighs the diversification effect. If both firms engage in M&A activity this leads to a higher wealth only if $Z_0 > Z_2$. That is, only if firm 1 becomes less profitable relative to firm 2. Since the numerical example is set-up such that in all cases $Z_1 < \underline{Z} < 1 < \bar{Z} < Z_2$, this indicates that if $Z_0 < Z_1$ the investor prefers an *early* merger, whereas for Z_0 the investor prefers a *later* merger.

In short, Figure 2 suggests that with low correlation, even high synergies (around 60%) do not make M&A activity the preferred option by the investor. For high ρ, even lower synergies make mergers the preferred option. However, the investor is ambiguous as to early or late mergers. These results do indicate that investors prefer mergers between related firms, when it is reasonable to assume a high value of ρ. For example, the merger wave of the 1970s which was characterised by high degrees of diversification and low synergies was, by this reasoning, not optimal for risk averse shareholders.

Appendix

A Proof of Lemma 1

Applying Ito's lemma we find that

$$\begin{aligned}
dY =&\gamma X_1^{\gamma-1}X_2^{1-\gamma}dX_1 + (1-\gamma)X_1^{\gamma}X_2^{-\gamma}dX_2 \\
&-\frac{1}{2}\gamma(1-\gamma)\Big(X_1^{\gamma-2}X_2^{1-\gamma}(dX_1)^2 + X_1^{\gamma}X_2^{-\gamma-1}(dX_2)^2\Big) \\
&+\gamma(1-\gamma)X_1^{\gamma-1}X_2^{-\gamma}dX_1dX_2 \\
=&\gamma\mu_1 Ydt + \gamma\sigma_1 YdW_1 + (1-\gamma)\mu_2 Ydt + (1-\gamma)\sigma_2 YdW_2 \\
&-\frac{1}{2}\gamma(1-\gamma)(\sigma_1^2+\sigma_2^2)Ydt + \gamma(1-\gamma)\sigma_1\sigma_2\rho Ydt \\
\equiv&\mu_Y dt + \gamma\sigma_1 YdW_1 + (1-\gamma)\sigma_2 YdW_2,
\end{aligned}$$

where μ_Y is defined as in (3).

Consider $d\tilde{W}_1 \equiv \gamma\sigma_1 dW_1 \sim \mathcal{N}(0, \gamma^2\sigma_1^2 dt)$ and $d\tilde{W}_2 \equiv (1-\gamma)\sigma_2 dW_2 \sim \mathcal{N}(0, (1-\gamma)^2\sigma_2^2 dt)$. Note that $Cov(dW_1, dW_2) = \mathbb{E}(dW_1 dW_2) = \rho dt$. Hence, it holds that

$$Cov(d\tilde{W}_1, d\tilde{W}_2) = \gamma(1-\gamma)\sigma_1\sigma_2\rho dt.$$

Let W^Y be a Wiener process such that $dW^Y = d\tilde{W}_1 + d\tilde{W}_2$. Then it holds that

$$dW^Y \sim \mathcal{N}(0, [\gamma^2\sigma_1^2 + (1-\gamma)^2\sigma_2^2 + 2\gamma(1-\gamma)\sigma_1\sigma_2\rho]dt).$$

Note that

$$\gamma\sigma_1 YdW_1 + (1-\gamma)\sigma_2 YdW_2 = \sigma_Y YdW^Y,$$

where and σ_Y is as defined in (4). The proof for $(Z_t)_{t\geq 0}$ follows along the same lines.

□

B Optimal Stopping Theory

Let $(X_t)_{t\geq 0}$ be an Ito diffusion on a domain $V \subset \mathbb{R}^n$, defined by

$$dX_t = b(Y_t)dt + \sigma(Y_t)dB_t,$$

with $dB_i dB_j = \rho_{ij}dt$ and $\rho_{ii} = 1$, for all $i = 1, \ldots, n$. A time-homogenous optimal stopping problem on (X_t), with *reward function* $g : V \to \mathbb{R}_+$ and *instantaneous reward function* $f : V \to \mathbb{R}$, is of the form: Find (g^*, τ^*) such that

$$g^*(x) = \sup_\tau \mathbb{E}\Big[\int_0^\tau f(X_t)dt + g(X_\tau)\Big] = \mathbb{E}\Big[\int_0^{\tau^*} f(X_t)dt + g(X_{\tau^*})\Big], \tag{B.1}$$

the supremum being taken over all stopping times τ for (X_t). Define

$$T = \sup\{t > 0 | X_t \notin V\}.$$

Furthermore, define the the *partial differential operator* $\mathcal{L}_X$,

$$\mathcal{L}_X = \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma\sigma')_{ij}(y) \rho_{ij} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Consider a function $\varphi : \bar{V} \to \mathbb{R}$ and the set $D = \{x \in V | \varphi(x) > g(x)\}$. The following theorem is from Øksendal (2000, p. 213).

Theorem B.1 (Variational inequalities for optimal stopping) *If the following conditions hold:*

1. $\varphi \in C^1(V) \cap C(\bar{V})$*;*
2. $\varphi \geq g$ *on* V *and* $\varphi = g$ *on* ∂V*;*
3. $\mathbb{E} \int_0^T \mathbb{1}_{\partial D}(X_t) dt = 0$*;*
4. ∂D *is a Lipschitz surface;*
5. $\varphi \in C^2(V \backslash \partial D)$ *and the second order derivatives of* φ *are locally bounded near* ∂D*;*
6. $\mathcal{L}_X \varphi + f \leq 0$ *on* $V \backslash \bar{D}$*;*
7. $\mathcal{L}_X \varphi + f = 0$ *on* D*;*
8. $\tau_D := \inf\{t > 0 | X_t \notin D\} < \infty$ *a.s.;*
9. *the family* $\{\varphi(X_\tau) | \tau \leq \tau_D\}$ *is uniformly integrable w.r.t. the probability law of* X_t*.*

Then $g^*(x) = \varphi(x) = \sup_{\tau \leq T} \mathbb{E}\Big[\int_0^\tau f(X_t)dt + g(X_\tau) \Big]$*, and* $\tau^* = \tau_D$*, solve the optimal stopping problem (B.1).*

C Proof of Lemma 2

Define the functions $f_1(z) = A_1 z^{\beta_1}$, $f_2(z) = A_2 z^{\beta_2}$, and $g(z) = \frac{D_m}{r-\mu_Y} z^\gamma - \frac{D_1}{r-\mu_1} z - \frac{D_2}{r-\mu_2}$. Applying the first and second order conditions yields that g has a global maximum at

$$\bar{Z} = \left(\gamma \frac{D_m}{D_1} \frac{r - \mu_1}{r - \mu_Y} \right)^{\frac{1}{1-\gamma}}.$$

Under condition (12) it holds that $g(\bar{Z}) > 0$. Given that $f_1'(z) > 0$ and $f_2'(z) < 0$ this immediately leads to the desired result. See also Figure 3 . □

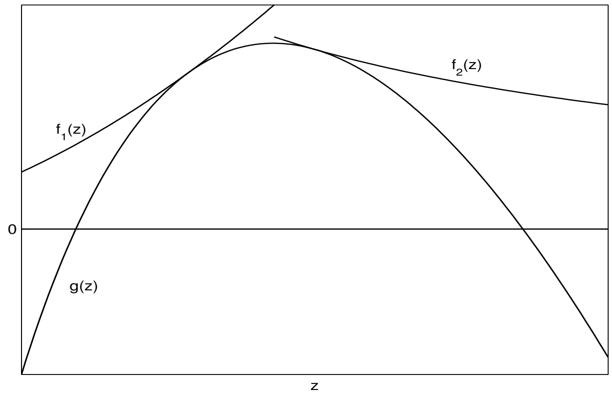

Figure 3: Graph of the functions f_1, f_2, and g.

D Proof of Lemma 4

Note that

$$\begin{aligned} L_1(x_1,x_2) \geq F_1(x_1,x_2) &\iff \frac{D_m}{r-\mu_Y}z^\gamma - \frac{D_2}{r-\mu_2} \geq \frac{D_1}{r-\mu_1}z \\ &\iff g_1(z) \equiv \frac{D_m}{r-\mu_Y}z^\gamma - \frac{D_1}{r-\mu_1}z - \frac{D_2}{r-\mu_2} \geq 0. \end{aligned}$$

from the proof of Lemma 2 we know that under (12), the function g_1 has a global maximum at, say, z^*, with $g_1(z^*) > 0$. Since g_1 is strictly concave, this implies that there exist Z_1 and $Z_2 > Z_1$ such that $L_1(z) \geq F_1(z) \iff Z_1 \leq z \leq Z_2$.

Furthermore, it holds that

$$L_2(x_1,x_2) \geq F_2(x_1,x_2) \iff g_2(z) \equiv \frac{D_m}{r-\mu_Y}(1/z)^{1-\gamma} - \frac{D_2}{r-\mu_2}(1/z) - \frac{D_1}{r-\mu_1} \geq 0.$$

Since $g_2(z) = zg_1(z)$, g_2 has the same zeros on $\mathbb{R}_{++}$. $\square$

E Proof of Lemma 5

The proof is an extension to two-dimensional processes of Dixit and Pindyck (1994, Appendix 9.A). Let $z < Z^*$. Note, first, that $x_1 \to x_2 Z^* \Rightarrow z \to Z^*$. Define

$$f(x_1,x_2) = \mathbb{E}\left(e^{-rT^*}\right).$$

If $\frac{X_{1t}}{X_{2t}} < Z^*$, then it holds that $\frac{X_{1,t+dt}}{X_{2,t+dt}} < Z^*$ a.s. Hence, $f(\cdot)$ satisfies the Bellman equation

$$f(x_1,x_2) = e^{-rdt}\mathbb{E}\big(f(x_1+dX_1, x_2+dX_2)\big).$$

Using a Taylor series expansion around $dt = 0$, we get

$$f(x_1, x_2) = (1 - rdt + o(dt))\big[\mathbb{E}(df) + f\big].$$

After applying Ito's lemma, substituting $f(x_1, x_2) = x_2\varphi(z)$, rearranging and taking $dt \downarrow 0$, this leads to the PDE

$$0 = \frac{1}{2}\sigma_Z^2 z^2\varphi''(z) + (\mu_1 - \mu_2)z\varphi'(z) - (r - \mu_2)\varphi(z),$$

which has as general solution

$$f(x_1, x_2) = A_1\Big(\frac{x_1}{x_2}\Big)^{\beta_1} + A_2\Big(\frac{x_1}{x_2}\Big)^{\beta_2}.$$

If $x_1 \downarrow 0$, it holds that $z \downarrow 0$, and, hence, that $T^* \to \infty$, and $e^{-rT^*} \downarrow 0$. Since $\beta_2 < 0$, it should, therefore, hold that $A_2 = 0$. Furthermore, if $x_1 \to x_2Z^*$, it holds that $T^* \downarrow 0$, and, hence, that $e^{-rT^*} \to 1$. From the condition $f(x_2Z^*, x_2) = 1$ one obtains $A_1 = \frac{1}{x_2}(Z^*)^{-\beta_1}$. Substituting yields $f(x_1, x_2) = (z/Z^*)^{\beta_1}$. The proof for $z > Z^*$ follows along similar lines. □

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