Some properties of profit functions and supply functions

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Differentiable technology, the curvature of the profit function,

and the response of supply to own-price changes

Abstract: This note begins by establishing a property of net supply for a competitive firm: assuming differentiability of the production frontier, linearly independent price-vectors have disjoint image-sets under the supply mapping. This supports the main results: first, a simple proof of McFadden's proposition that differentiability of the production frontier is necessary and sufficient for the profit function to be strictly quasi-convex; and secondly, a proof that for discrete price-changes, own-price effects in supply are strictly positive, assuming a differetiable technology. Finally, the implications for cost and demand theory are indicated.

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INTRODUCTION

Profit, cost and expenditure functions are widely used in pure and applied economics. For example, these keys produce a total of 1,100 hits in the *Econlit* database in September 2003. A competitive firm's profit function is convex in prices, while cost and expenditure functions are concave. For differentiable functions, these properties, with Hotelling's or Shephard's lemmas, imply weak-inequality own-price effects on net supplies or demands; for discrete price-changes, similar results flow directly from the logic of optimality. Many years ago, McFadden (1978a, pp. 34-35, 89-90, and appendices) showed that cost and profit functions for differentiable technologies are in fact *strictly* quasi-concave and *strictly* quasi-convex respectively, so that between any pair of linearly independent price-vectors, the arc of the function is correspondingly *strictly* concave or *strictly* convex. This has not been widely reported, the proofs are difficult, and standard sources present only weak convexity or concavity and weak-form own-price effects: for example, Barten and Böhm (1982, p. 402), Gravelle and Rees (1992, pp. 204, 209, 242-244), Kreps (1990, pp. 48, 244, 251), Mas-Collel *et al*. (1995, pp. 59, 63, 138, 141) and Varian (1992, pp. 36, 41, 61, 72, 105). A partial exception is Nadiri (1982, p. 437).[1]

We begin with an extremely simple alternative proof of McFadden's result for profits. We then derive strong-form own-price effects in supply for discrete price-changes. Finally we present conclusions, with extensions to costs and demand.

STRICT QUASI CONVEXITY OF THE PROFIT FUNCTION

Preliminary lemma

Given a production set $\boldsymbol{Y} \subset R^N$ and any price-vector $\boldsymbol{p}>\mathbf{0}$, a competitive firm chooses $\boldsymbol{y} \in \boldsymbol{Y}$ to maximize profits $\boldsymbol{p.y}$. Maxima exist if $\boldsymbol{Y}$ is closed and if unbounded actions are impossible,[2] and the solutions generate the supply correspondence $\boldsymbol{Y(p)}$ and the profit function $\pi(\boldsymbol{p})$.

Let $\boldsymbol{Y}$ be representable implicitly by a differentiable function $g(\boldsymbol{y})$, with $g(\boldsymbol{y})=0$ on the frontier. The first-order conditions reduce to: $\frac{p_i}{p_j} = \frac{\partial g}{\partial y_i} / \frac{\partial g}{\partial y_j} = M_{ji}$,[3] $\forall i,j,\ i \neq j$, unless a boundary condition binds on i or j.[4] If $\boldsymbol{p}'$ and $\boldsymbol{p}''$ are linearly independent, then for some i,j, $\frac{p_i'}{p_j'} \neq \frac{p_i''}{p_j''}$, so that a given point $\boldsymbol{y}$ can never be optimal at both $\boldsymbol{p}'$ and $\boldsymbol{p}''$, because $M_{ji}(\boldsymbol{y})$ cannot equal both $\frac{p_i'}{p_j'}$ and $\frac{p_i''}{p_j''}$. Formally, $M_{ji}(\boldsymbol{y}') \neq M_{ji}(\boldsymbol{y}'')$ for any $\boldsymbol{y}' \in \boldsymbol{Y(p')}$ and $\boldsymbol{y}'' \in \boldsymbol{Y(p'')}$, and we have:

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[1] See note 5 below on Nadiri.

[2] For convex technologies, with additional restrictions on production sets, no firm has access to unbounded actions (Starr, 1997, pp. 112-114). Alternatively, assume the production set to be semi-bounded, i.e. that there exists a non-empty set of prices at which solutions exist to the profit maximization problem (McFadden, 1978a, p. 62).

[3] M_{ji} is a marginal product, or its inverse, or a marginal rate of transformation or substitution.

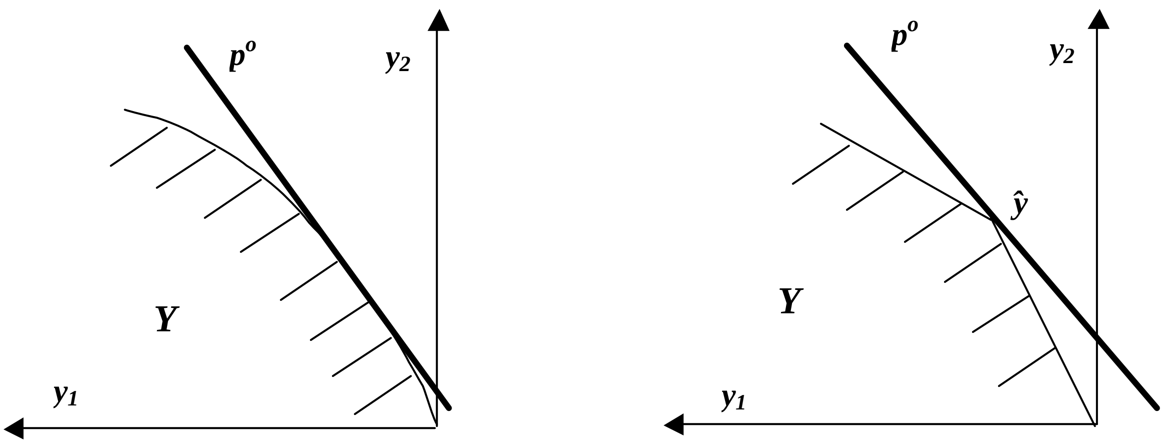
[4] Corner solutions are ignored, but these involve points of non-differentiability in the frontier of the opportunity set, i.e. the intersection of $\boldsymbol{Y}$ with the set defined by boundary conditions.

Lemma 1 *Let **p**' and **p**" be any pair of linearly independent price-vectors. If g(**y**) is differentiable, then **Y(p')**∩**Y(p")**=Æ. If **Y(p)** is a function, it is one-to-one.*

FIGURE 1. Production sets illustrating:

(a) differentiability with weak convexity;

(b) a point of non-differentiability in the production frontier.



(a)

Differentiability, without uniqueness: points on the linear segment of the frontier cannot be optimal at price-vectors that are linearly independent of $\boldsymbol{p}^o$.

(b)

Non-differentiability: $\hat{\boldsymbol{y}}$ is optimal, not just at $\boldsymbol{p}^o$, but on a convex set of price-vectors including $\boldsymbol{p}^o$.

Figure 1 illustrates this result, for technologies where good 1 is the sole input to the production of good 2. In Figure 1(a) multiple optima exist at prices $\boldsymbol{p}^o$ on a linear and differentiable segment of the frontier. None of these points could be optimal at a price-vector that was not a scalar multiple of $\boldsymbol{p}^o$. At $\hat{\boldsymbol{y}}$ in Figure 1(b), differentiability is violated, M_{21} is undefined, and $\hat{\boldsymbol{y}}$ is optimal on a convex set of prices that includes $\boldsymbol{p}^o$.

The main result

Differentiability of $g(\boldsymbol{y})$ is sufficient and necessary for strict quasi-convexity of $p(\boldsymbol{p})$: i.e.,

$\forall\ \boldsymbol{p}', \boldsymbol{p}'' \in R^{N+}, \boldsymbol{p}' \neq \boldsymbol{p}''$, and $\forall t \in (0,1)$: $p(\boldsymbol{p}^t) < \max\{p(\boldsymbol{p}'), p(\boldsymbol{p}'')\}$, where $\boldsymbol{p}^t = t\boldsymbol{p}' + (1-t)\boldsymbol{p}''$.

Sufficiency. Let $g(\boldsymbol{y})$ be differentiable. First, let $\boldsymbol{p}'$ and $\boldsymbol{p}''$ be linearly independent, and define $\boldsymbol{p}^t = t\boldsymbol{p}' + (1-t)\boldsymbol{p}''$ for any $t \in (0,1)$. Let $\boldsymbol{y}^t \in Y(\boldsymbol{p}^t)$. As $\boldsymbol{p}^t$ is linearly independent of $\boldsymbol{p}'$ and $\boldsymbol{p}''$, Lemma 1 applies, so $\boldsymbol{y}^t \notin \boldsymbol{Y}(\boldsymbol{p}')$ and $\boldsymbol{y}^t \notin \boldsymbol{Y}(\boldsymbol{p}'')$: therefore $\boldsymbol{p}'.\boldsymbol{y}^t < p(\boldsymbol{p}')$ and $\boldsymbol{p}''.\boldsymbol{y}^t < p(\boldsymbol{p}'')$. Taking a convex combination, we immediately have $p(\boldsymbol{p})$ strictly convex on $[\boldsymbol{p}', \boldsymbol{p}'']$.[5]

Alternatively, let $\boldsymbol{p}'' = k\boldsymbol{p}'$, some $k > 0$, $k \neq 1$. Then $\boldsymbol{p}^t = t\boldsymbol{p}' + k(1-t)\boldsymbol{p}' = q\boldsymbol{p}'$, and $\boldsymbol{Y}(\boldsymbol{p}') = \boldsymbol{Y}(\boldsymbol{p}'') = \boldsymbol{Y}(\boldsymbol{p}^t)$, by zero-homogeneity of $\boldsymbol{Y}(\boldsymbol{p})$. Using linear homogeneity of $p(\boldsymbol{p})$, $p(\boldsymbol{p}^t)$ lies strictly be tween $p(\boldsymbol{p}')$ and $p(\boldsymbol{p}'')$, which are unequal because $k \neq 1$, and $p(\boldsymbol{p}^t) < \max\{p(\boldsymbol{p}'), p(\boldsymbol{p}'')\}$.[6]

Necessity. Let $p(\boldsymbol{p})$ be strictly quasi-convex. If $g(\boldsymbol{y})$ is not differentiable at $\hat{\boldsymbol{y}}$ Î $\boldsymbol{Y}$ then Lemma 1 is inapplicable at $\hat{\boldsymbol{y}}$. Suppose that $\hat{\boldsymbol{y}}$ were optimal at some $\boldsymbol{p}$, which would generally not be unique. All such $\boldsymbol{p}$ would form a convex set $C^{\hat{y}}$ on which $p(\boldsymbol{p})$ would be linear: $p(\boldsymbol{p}) = \hat{\boldsymbol{y}}.\boldsymbol{p}$, all $\boldsymbol{p} \in C^{\hat{y}}$, violating strict quasi-convexity.[7] Thus at any $\boldsymbol{p}^{\mathrm{o}} \in R^{N+}$, strict quasi-convexity of $p(\boldsymbol{p})$ in the neighbourhood of $\boldsymbol{p}^{\mathrm{o}}$ must imply differentiability of $g(\boldsymbol{y})$ at all $\boldsymbol{y} \in \boldsymbol{Y}(\boldsymbol{p}^{\mathrm{o}})$.

[5] Strict convexity implies strict quasi-convexity, but the converse does not hold. A version of the result in the text is stated, without explicit reference to differentiability, in Nadiri (1982, 437). Nadiri (p. 451) lists some of McFadden's results.

[6] Let $k<1$, so $q<1$, and $p(\boldsymbol{p}^t) = p(q\boldsymbol{p}') = qp(\boldsymbol{p}') < p(\boldsymbol{p}') = \max\{p(\boldsymbol{p}'), p(\boldsymbol{p}'')\}$, $p(\boldsymbol{p})$ being homogeneous of degree one. For $k>1$, $p(\boldsymbol{p}^t) < \max\{p(\boldsymbol{p}'), p(\boldsymbol{p}'')\}$ also, as the labelling of $\boldsymbol{p}'$ and $\boldsymbol{p}''$ is arbitrary.

[7] For any linearly independent $\boldsymbol{p}'$ and $\boldsymbol{p}''$ in $C^{\hat{y}}$, it is always possible to normalize $\boldsymbol{p}''$ so that $\boldsymbol{p}'.\hat{\boldsymbol{y}} = \boldsymbol{p}''.\hat{\boldsymbol{y}} = \boldsymbol{p}^t.\hat{\boldsymbol{y}}$, where $\boldsymbol{p}^t$ is a convex combination of $\boldsymbol{p}'$ and $\boldsymbol{p}''$, and then $p(\boldsymbol{p}^t) = \max\{p(\boldsymbol{p}'), p(\boldsymbol{p}'')\}$.

THE IMPACT OF A DISCRETE OWN-PRICE CHANGE

If $\pi(\boldsymbol{p})$ is twice differentiable, Hotelling's lemma and convexity of $\pi(\boldsymbol{p})$ together imply $\frac{\partial y_i(\boldsymbol{p})}{\partial p_i} \geq 0$, each i. To strengthen this we may make further assumptions about the Hessian of $\pi(\boldsymbol{p})$; alternatively, we may show that arbitrarily close to any $\boldsymbol{p}^o$ there exist points at which $\frac{\partial y_i(\boldsymbol{p})}{\partial p_i} > 0$, each i (McFadden, 1978b, p. 403; Takayama, 1994, p. 141). However, Lemma 1 facilitates a stronger result, without calculus, except for assuming differentiability of $g(\boldsymbol{y})$.

Given $\boldsymbol{y}' \in \boldsymbol{Y}(\boldsymbol{p}')$ and $\boldsymbol{y}'' \in \boldsymbol{Y}(\boldsymbol{p}'')$, $(\boldsymbol{p}'-\boldsymbol{p}'').(\boldsymbol{y}'-\boldsymbol{y}'') \geq 0$ (Varian, 1992, p. 36). Choose $p_j'=p_j''$, all $j \neq i$. Then for any p_i' and p_i'', $(p_i'-p_i'')(y_i'-y_i'') \geq 0$. If $p_i' \neq p_i''$, $\boldsymbol{p}'$ and $\boldsymbol{p}''$ are linearly independent, assuming $p_j' \neq 0$, some $j \neq i$, and Lemma 1 applies, given $g(\boldsymbol{y})$ differentiable. Both inequalities are then strict, and the effect of an own-price change on net output is strictly positive, *cet. par*.

CONCLUSIONS

If $g(\boldsymbol{y})$ is differentiable then linearly independent price-vectors have disjoint image-sets under the supply mapping. Consequently, $\pi(\boldsymbol{p})$ is strictly quasi-convex in the neighbourhood of any $\boldsymbol{p}^o \in R^{N+}$ if and only if $g(\boldsymbol{y})$ is differentiable at each $\boldsymbol{y} \in \boldsymbol{Y}(\boldsymbol{p}^o)$. For $g(\boldsymbol{y})$ differentiable, own-price effects on net supply are strictly positive, whether or not $\pi(\boldsymbol{p})$ is differentiable.

Strict convexity of $\boldsymbol{Y}$ is neither necessary nor sufficient for these results. For $\boldsymbol{Y}$ strictly convex, $\boldsymbol{Y}(\boldsymbol{p})$ is single-valued, but the results are driven by the disjointedness of $\boldsymbol{Y}(\boldsymbol{p}')$, $\boldsymbol{Y}(\boldsymbol{p}'')$ and $\boldsymbol{Y}(\boldsymbol{p}^t)$, not by the multiplicity of the solutions at each $\boldsymbol{p}$.

Lemma 1 adapts naturally to cost (or consumer) theory: for a differentiable isoquant (indifference) surface, the cost (expenditure) function is strictly quasi-concave, and conditional input-demands (compensated demands) have strictly negative discrete own-price effects.

REFERENCES

Barten, A.P. and V. Böhm. 1982. Consumer theory. In: *Handbook of mathematical economics*, vol. II, ed. K.J. Arrow and M.D. Intriligator. Amsterdam: North-Holland. Pp. 381-429.

Gravelle, H. and R. Rees. 1992. *Microeconomics*. 2nd ed. London: Longman.

Kreps, D. M. 1990. *A course in microeconomic theory*. New York: Harvester Wheatsheaf.

McFadden, D. 1978a. Cost, revenue, and profit functions. In: *Production economics: a dual approach to theory and applications*, vol. 1, ed. M. Fuss and D. McFadden. Amsterdam: North Holland. Pp. 3-109.

McFadden, D. 1978b. Convex analysis. In: *Production economics: a dual approach to theory and applications*, Vol. 1, ed. M. Fuss and D. McFadden. Amsterdam: North Holland. Pp. 383-408.

Mas-Collel, A., Whinston, M.D. and Green, J.R. 1995. *Microeconomic analysis*. Oxford: Oxford University Press.

Nadiri, M.I. 1982. Producers' theory. In: *Handbook of mathematical economics*, vol. II, ed. K.J. Arrow and M.D. Intriligator. Amsterdam: North-Holland. Pp. 431-490.

Starr, R.M. 1997. *General equilibrium theory*. Cambridge: Cambridge University Press.

Takayama, A. 1994. *Analytical methods in economics*. New York: Harvester Wheatsheaf.

Varian, H.R. 1992. *Microeconomic analysis*. 3rd ed. New York: Norton.