Some properties of profit functions and supply functions

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by

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Differentiable technology, the curvature of the profit function, and the response of supply to own-price changes

Abstract: This note begins by establishing a property of net supply for a competitive firm: assuming differentiability of the production frontier, linearly independent price-vectors have disjoint image-sets under the supply mapping. This supports the main results: first, a simple proof of McFadden's proposition that differentiability of the production frontier is necessary and sufficient for the profit function to be strictly quasi-convex; and secondly, a proof that for discrete price-changes, own-price effects in supply are strictly positive, assuming a differentiable technology. Finally, the implications for cost and demand theory are indicated.

Keywords: Profit function; convexity; strict quasi-convexity; Hotelling's lemma; cost function; expenditure function; differentiability; own-price effect.

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INTRODUCTION

Profit, cost and expenditure functions are widely used in pure and applied economics. For example, these keys produce a total of 1,100 hits in the Econlit database in September 2003. A competitive firm's profit function is convex in prices, while cost and expenditure functions are concave. For differentiable functions, these properties, with Hotelling's or Shephard's lemmas, imply weak-inequality own-price effects on net supplies or demands; for discrete price-changes, similar results flow directly from the logic of optimality. Many years ago, McFadden (1978a, pp. 34-35, 89-90, and appendices) showed that cost and profit functions for differentiable technologies are in fact strictly quasi-concave and strictly quasi-convex respectively, so that between any pair of linearly independent price-vectors, the arc of the function is correspondingly strictly concave or strictly convex. This has not been widely reported,
the proofs are difficult, and standard sources present only weak convexity or concavity and weak-form own-price effects: for example, Barten and Böhm (1982, p. 402), Gravelle and Rees (1992, pp. 204, 209, 242-244), Kreps (1990, pp. 48, 244, 251), Mas-Collel et al. (1995, pp. 59, 63, 138, 141) and Varian (1992, pp. 36, 41, 61, 72, 105). A partial exception is Nadiri (1982, p. 437).1

We begin with an extremely simple alternative proof of McFadden's result for profits. We then derive strong-form own-price effects in supply for discrete price-changes. Finally we present conclusions, with extensions to costs and demand.

**STRICT QUASI CONVEXITY OF THE PROFIT FUNCTION**

**Preliminary lemma**

Given a production set $Y \subset \mathbb{R}^N$ and any price-vector $p > 0$, a competitive firm chooses $y \in Y$ to maximize profits $p_y$. Maxima exist if $Y$ is closed and if unbounded actions are impossible,2 and the solutions generate the supply correspondence $Y(p)$ and the profit function $\pi(p)$.

Let $Y$ be representable implicitly by a differentiable function $g(y)$, with $g(y) = 0$ on the frontier. The first-order conditions reduce to: $\frac{\partial g}{\partial y_i} \frac{\partial g}{\partial y_j} = M_{ji}, \forall i, j, i \neq j$, unless a boundary condition binds on $i$ or $j$.3 If $p'$ and $p''$ are linearly independent, then for some $i,j, \frac{p'_i}{p'_j} \neq \frac{p''_i}{p''_j}$, so that a given point $y$ can never be optimal at both $p'$ and $p''$, because $M_{ji}(y)$ cannot equal both $\frac{p'_i}{p'_j}$ and $\frac{p''_i}{p''_j}$.

Formally, $M_{ji}(y') \neq M_{ji}(y'')$ for any $y' \in Y(p')$ and $y'' \in Y(p'')$, and we have:

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1 See note 5 below on Nadiri.
2 For convex technologies, with additional restrictions on production sets, no firm has access to unbounded actions (Starr, 1997, pp. 112-114). Alternatively, assume the production set to be semi-bound ed, i.e. that there exists a non-empty set of prices at which solutions exist to the profit maximization problem (McFadden, 1978a, p. 62).
3 $M_{ji}$ is a marginal product, or its inverse, or a marginal rate of transformation or substitution.
4 Corner solutions are ignored, but these involve points of non-differentiability in the frontier of the opportunity set, i.e. the intersection of $Y$ with the set defined by boundary conditions.
Lemma 1  Let \( p' \) and \( p'' \) be any pair of linearly independent price-vectors. If \( g(y) \) is differentiable, then \( Y(p') \cap Y(p'') = \emptyset \). If \( Y(p) \) is a function, it is one-to-one.

**FIGURE 1.** Production sets illustrating:

(a) differentiability with weak convexity;

(b) a point of non-differentiability in the production frontier.

Figure 1 illustrates this result, for technologies where good 1 is the sole input to the production of good 2. In Figure 1(a) multiple optima exist at prices \( p^o \) on a linear and differentiable segment of the frontier. None of these points could be optimal at a price-vector that was not a scalar multiple of \( p^o \). At \( \hat{y} \) in Figure 1(b), differentiability is violated, \( M_{21} \) is undefined, and \( \hat{y} \) is optimal on a convex set of prices that includes \( p^o \).
The main result

Differentiability of $g(y)$ is sufficient and necessary for strict quasi-convexity of $\pi(p)$: i.e.,

$$\forall p', p^\prime \in R^N, p^\prime \neq p^\prime, \text{ and } \forall t \in (0,1): \pi(p^t) < \max\{\pi(p'), \pi(p^\prime)\},$$

where $p^t = tp + (1-t)p^\prime$.

Sufficiency. Let $g(y)$ be differentiable. First, let $p'$ and $p''$ be linearly independent, and define $p^t = tp + (1-t)p''$ for any $t \in (0,1)$. Let $y' \in Y(p')$. As $p'$ is linearly independent of $p'$ and $p''$, Lemma 1 applies, so $y' \notin Y(p')$ and $y' \notin Y(p'')$: therefore $p', y' < \pi(p')$ and $p'', y' < \pi(p'')$. Taking a convex combination, we immediately have $\pi(p)$ strictly convex on $[p', p'']$.

Alternatively, let $p'' = kp'$, some $k > 0$, $k \neq 1$. Then $p' = p' + k(1-t)p' = 0p'$, and $Y(p') = Y(p'') = Y(p')$, by zero-homogeneity of $Y(p)$. Using linear homogeneity of $\pi(p)$, $\pi(p')$ lies strictly between $\pi(p')$ and $\pi(p'')$, which are unequal because $k \neq 1$, and $\pi(p') < \pi(p') \leq \max\{\pi(p'), \pi(p'')\}$.

Necessity. Let $\pi(p)$ be strictly quasi-convex. If $g(y)$ is not differentiable at $y \in Y$ then Lemma 1 is inapplicable at $y$. Suppose that $y'$ were optimal at some $p$, which would generally not be unique. All such $p$ would form a convex set $C^p$ on which $\pi(p)$ would be linear: $\pi(p) = \lambda p$, all $p \in C^p$, violating strict quasi-convexity. Thus at any $p^0 \in R^N$, strict quasi-convexity of $\pi(p)$ in the neighbourhood of $p^0$ must imply differentiability of $g(y)$ at all $y \in Y(p^0)$.

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5 Strict convexity implies strict quasi-convexity, but the converse does not hold. A version of the result in the text is stated, without explicit reference to differentiability, in Nadiri (1982, 437). Nadiri (p. 451) lists some of McFadden’s results.

6 Let $k \leq 1$, so $0 < 1$, and $\pi(p') = \pi(p') = \pi(p') = \max\{\pi(p'), \pi(p')\}, \pi(p)$ being homogeneous of degree one. For $k > 1$, $\pi(p') \leq \pi(p') \leq \max\{\pi(p'), \pi(p')\}$ also, as the labelling of $p'$ and $p''$ is arbitrary.

7 For any linearly independent $p'$ and $p''$ in $C^p$, it is always possible to normalize $p''$ so that $p^t \neq p^t \neq p^t \neq p^t$, where $p^t$ is a convex combination of $p'$ and $p''$, and then $\pi(p^t) = \max\{\pi(p'), \pi(p'')\}$.
THE IMPACT OF A DISCRETE OWN-PRICE CHANGE

If $\pi(p)$ is twice differentiable, Hotelling's lemma and convexity of $\pi(p)$ together imply
\[ \frac{\partial y_i(p)}{\partial p_i} \geq 0, \text{ each } i. \]
To strengthen this we may make further assumptions about the Hessian of $\pi(p)$; alternatively, we may show that arbitrarily close to any $p^o$ there exist points at which
\[ \frac{\partial y_i(p)}{\partial p_i} > 0, \text{ each } i \] (McFadden, 1978b, p. 403; Takayama, 1994, p. 141). However, Lemma 1 facilitates a stronger result, without calculus, except for assuming differentiability of $g(y)$.

Given $y' \in Y(p')$ and $y'' \in Y(p'')$, $(p' - p'')(y' - y'') \geq 0$ (Varian, 1992, p. 36). Choose $p_j = p_j'$, all $j \neq i$. Then for any $p_i$ and $p_i''$, $(p' - p)(y' - y) \geq 0$. If $p_i \neq p_i''$, $p'$ and $p''$ are linearly independent, assuming $p_j \neq 0$, some $j \neq i$, and Lemma 1 applies, given $g(y)$ differentiable. Both inequalities are then strict, and the effect of an own-price change on net output is strictly positive, *cet. par.*

CONCLUSIONS

If $g(y)$ is differentiable then linearly independent price-vectors have disjoint image-sets under the supply mapping. Consequently, $\pi(p)$ is strictly quasi-convex in the neighbourhood of any $p^o \in R^N_+$ if and only if $g(y)$ is differentiable at each $y \in Y(p^o)$. For $g(y)$ differentiable, own-price effects on net supply are strictly positive, whether or not $\pi(p)$ is differentiable.

Strict convexity of $Y$ is neither necessary nor sufficient for these results. For $Y$ strictly convex, $Y(p)$ is single-valued, but the results are driven by the disjointedness of $Y(p')$, $Y(p'')$ and $Y(p')$, not by the multiplicity of the solutions at each $p$.

Lemma 1 adapts naturally to cost (or consumer) theory: for a differentiable isoquant (indifference) surface, the cost (expenditure) function is strictly quasi-concave, and conditional input-demands (compensated demands) have strictly negative discrete own-price effects.
REFERENCES


