Existence of Equilibrium and Price Adjustments in a Finance Economy with Incomplete Markets*

Dolf Talman† Jacco Thijssen‡

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Abstract

In this paper the standard two-period general equilibrium model with incomplete financial markets is considered. First, existence of equilibrium is proved using a stationary point argument on the set of no-arbitrage prices. Prices are normalized with respect to the market portfolio. The proof does not use the commonly applied normalization on the unit sphere or truncation of the set of prices. Also a new price adjustment process is proposed. The process generates a path of price vectors from an arbitrary price vector to an equilibrium. The path can be followed by a simplicial algorithm for finding stationary points on polyhedra.

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†Corresponding author. A.J.J. Talman, Department of Econometrics & Operations Research and CentER, Tilburg University, Tilburg, PO Box 90153, 5000 LE Tilburg, the Netherlands. Email: talman@uvt.nl.

‡J.J.J. Thijssen, Department of Economics, Trinity College Dublin, Dublin 2, Ireland. Email: Jacco.Thijssen@tcd.ie.

1 Introduction

The main focus of this paper is to describe a price-adjustment process in an economy with incomplete financial markets, that converges to an equilibrium price vector. It turns out that the simplicial algorithm for calculating stationary points of a continuous function on a polytope as developed by Talman and Yamamoto (1989) can be used to describe price formation on financial markets.

In this paper the simplest general equilibrium model with incomplete markets as is presented in e.g. Magill and Quinzii (1996) is considered. There are two periods of time (present and future), a finite number of possible future states, one consumption good and a number of financial securities that can be used to transfer income from the present to the future. For the consumption good there are spot markets, so at present one cannot trade the consumption good for the future. Financial markets are incomplete if not all possible income streams for present and future can be attained by trading in the assets available on the existing financial markets.

Existence of equilibrium in a two-period general equilibrium model with multiple consumption goods and (possibly) incomplete markets is proved in Geanakoplos and Polemarchakis (1986). They prove existence on the set of no-arbitrage prices. These are prices such that it is impossible to create a portfolio of assets which generates a non-negative income stream in the future and has non-positive costs at present. The proof uses a fixed point argument for functions on compact sets. For that, since the set of no-arbitrage prices can be unbounded, the proof of Geanakoplos and Polemarchakis (1986) uses a compact truncation of this set. In this paper we present an existence proof for the one consumption good model that uses a stationary point argument without truncating the set of no-arbitrage prices. Other existence proofs use some transformation of the underlying model. Hens (1991) for example, introduces an artificial asset to translate the present into the future. The approach taken by Hirsch et al. (1990) shows existence of equilibrium in a model with only state prices. Then it is argued that each equilibrium in the original model corresponds one-to-one to an equilibrium in state prices.

Given that an equilibrium exists the question arises how to compute one. There is a homotopy method introduced in Herings and Kubler (2002) that requires differentiability assumptions on the utility functions. In this paper we show that one can use the simplicial approach developed by Talman and Yamamoto (1989), which does not require additional assumptions to the ones needed to prove existence. The algorithm generates a piecewise linear path and approximately follows the piecewise smooth path of a price adjustment process. The latter process connects an arbitrarily chosen initial price vector with an equilibrium price vector. Note that the

Talman and Yamamoto (1989) algorithm is defined for functions on polytopes. The set of no-arbitrage prices for the model can, however, be an unbounded polyhedron. Therefore, we extend the algorithm of Talman and Yamamoto (1989) to unbounded polyhedra. In the literature there are simplicial algorithms for functions on possibly unbounded polyhedra, notably by Dai et al. (1991) and Dai and Talman (1993). These algorithms cannot be applied, however, since they assume pointedness of the polyhedron and affine functions, respectively.

The paper is organised as follows. In Section 2 the economic model is described. In Section 3 we prove the existence of equilibrium and in Section 4 we adapt the simplicial algorithm of Talman and Yamamoto (1989) for the two-period finance economy. In Section 5 the algorithm is presented in some detail and illustrated by means of a numerical example.

2 The Finance Economy

The General Equilibrium model with Incomplete markets (GEI) explicitly includes incomplete financial markets in a general equilibrium framework. In this paper the simplest version is used. It consists of two time periods, $t = 0, 1$, where $t = 0$ denotes the present and $t = 1$ denotes the future. At $t = 0$ the state of nature is known to be $s = 0$. The state of nature at $t = 1$ is unknown and denoted by $s \in \{1, 2, \ldots, S\}$. In the economy there are $I \in \mathbb{N}$ consumers, indexed by $i = 1, \ldots, I$. There is one consumption good that can be interpreted as income. A consumption plan for consumer $i \in \{1, \ldots, I\}$ is a vector $x^i \in \mathbb{R}^{S+1}_+$, where x^i_s gives the consumption level in state $s \in \{0, 1, \ldots, S\}$.[1]

Each consumer $i = 1, \ldots, I$, is characterised by a vector of initial endowments, $\omega^i \in \mathbb{R}^{S+1}_+$, and a utility function $u^i : \mathbb{R}^{S+1}_+ \to \mathbb{R}$. Denote aggregate initial endowments by $\omega = \sum_{i=1}^{I} \omega^i$. Regarding the initial endowments and utility functions we make the following assumptions.

Assumption 1 *The vector of aggregate initial endowments is strictly positive, i.e.* $\omega \in \mathbb{R}^{S+1}_{++}$.

Assumption 2 *For each agent* $i = 1, \ldots, I$, *the utility function,* u^i, *is continuous, strictly monotone and strictly quasi-concave on* $\mathbb{R}^{S+1}_+$.

Assumption 1 ensures that in each period and in each state of nature there is at least one agent who has a positive amount of the consumption good. Assumption 2 ensures that the consumer's demand is a continuous function.

[1] In general we denote for a vector $x \in \mathbb{R}^{S+1}$, $x = (x_0, \mathbf{x_1}) \in \mathbb{R} \times \mathbb{R}^S$ to separate x_0 in period $t = 0$ and $\mathbf{x_1} = (x_1, \ldots, x_S)$ in period $t = 1$.

It is assumed that the market for the consumption good is a spot market. The consumers can smoothen consumption by trading on the asset market. At the asset market, $J \in \mathbb{N}$ financial contracts are traded, indexed by $j = 1, \ldots, J$. The future payoffs of the assets are put together in a matrix

$$V = (V^1, \ldots, V^J) \in \mathbb{R}^{S \times J},$$

where V_s^j is the payoff of one unit of asset j in state s. The following assumption is made with respect to V.

Assumption 3 *There are no redundant assets, i.e. $rank(V) = J$.*

Assumption 3 can be made without loss of generality; if there are redundant assets then a no-arbitrage argument guarantees that its price is uniquely determined by the other assets. Let the market subspace be denoted by $\langle V \rangle = Span(V)$. That is, the market subspace consists of those income streams that can be generated by trading on the asset market. If $S = J$, the market subspace consists of all possible income streams, i.e. markets are complete. If $J < S$ there is idiosyncratic risk and markets are incomplete.

A *finance economy* is defined as a tuple $\mathcal{E} = \Big((u^i, \omega^i)_{i=1,\ldots,I}, V\Big)$. Given a finance economy $\mathcal{E}$, agent i can trade assets by buying a portfolio $z^i \in \mathbb{R}^J$ given the (row)vector of prices $q = (q_0, q_{\mathbf{1}}) \in \mathbb{R}^{J+1}$, where q_0 is the price for consumption in period $t = 0$ and $q_{\mathbf{1}} = (q_1, \ldots, q_J)$ is the vector of security prices with q_j the price of security j, $j = 1, \ldots, J$. Given a vector of prices $q = (q_0, q_{\mathbf{1}}) \in \mathbb{R}^{J+1}$, the budget set for agent $i = 1, \ldots, I$ is given by

$$B^i(q) = \Big\{ x \in \mathbb{R}_+^{S+1} \Big| \exists_{z \in \mathbf{R}^J} : q_0(x_0 - \omega_0^i) \leq -q_{\mathbf{1}} z, x_{\mathbf{1}} - \omega_{\mathbf{1}}^i = Vz \Big\}. \tag{1}$$

Given the asset payoff matrix V we will restrict attention to asset prices that generate no arbitrage opportunities, i.e. asset prices q such that there is no portfolio generating a semi-positive income stream. Such asset prices exclude the possibility of "free lunches". The importance of restricting ourselves to no-arbitrage prices becomes clear from the following well-known theorem (cf. Magill and Quinzii (1996)).

Theorem 1 *Let $\mathcal{E}$ be a finance economy satisfying Assumption 2. Then the following conditions are equivalent:*

1. *$q \in \mathbb{R}^{J+1}$ permits no arbitrage opportunities;*
2. *$\forall_{i=1,\ldots,I} : \arg\max\{u^i(x^i) | x^i \in B^i(q)\} \neq \emptyset$;*
3. *$\exists_{\pi \in \mathbf{R}_{++}^S} : q_{\mathbf{1}} = \pi V$;*

4. *$B^i(q)$ is compact for all $i = 1, \ldots, I$.*

The vector $\pi \in \mathrm{I\!R}^S_{++}$ can be interpreted as a vector of state prices. Condition 3 therefore states that a no-arbitrage price for security j equals the present value of security j given the vector of state prices π. As a consequence of this theorem, in the remainder we restrict ourselves to the set of no-arbitrage prices

$$Q = \{q \in \mathrm{I\!R}^{J+1} | q_0 > 0, \exists_{\pi \in \mathbf{R}^S_{++}} : q_{\mathbf{1}} = \pi V\}. \tag{2}$$

Under Assumption 2, Theorem 1 shows that the demand function $x^i(q)$, maximising agent i's utility function $u^i(x)$ on $B^i(q)$, is well-defined for all $i = 1, \ldots, I$, and all $q \in Q$. Since the budget correspondence $B^i : Q \to \mathrm{I\!R}^{S+1}_+$ is upper- and lower-semicontinuous, Berge's maximum theorem gives that $x^i(q)$ is continuous on Q. Furthermore, because the mapping $z^i \mapsto Vz^i + \omega^i_{\mathbf{1}}$ is continuous, one-to-one and onto, the security demand function, $z^i(q)$, determined by $Vz^i(q) = x^i_{\mathbf{1}}(q) - \omega^i_{\mathbf{1}}$, is a continuous function on Q.

Define the *excess demand* function $f : Q \to \mathrm{I\!R}^{J+1}$ by

$$f(q) = \big(f_0(q), f_{\mathbf{1}}(q)\big) = \Big(\sum_{i=1}^{I}(x^i_0(q) - \omega^i_0), \sum_{i=1}^{I} z^i(q)\Big).$$

Note that since there are no initial endowments of asset j, $j = 1, \ldots, J$, the excess demand for asset j is given by $\sum_{i=1}^{I} z^i_j(q)$. With respect to the excess demand function we can derive the following result.

Lemma 1 *Under Assumptions 1–3 the excess demand function $f : Q \to \mathrm{I\!R}^J$ satisfies the following properties:*

1. *continuity on Q;*

2. *homogeneity of degree 0;*

3. *$(f_0(q), Vf_{\mathbf{1}}(q)) \geq -\omega$ for all $q \in Q$;*

4. *for all $q \in Q$, $qf(q) = 0$ (Walras' law).*

The proof of this lemma is elementary and therefore omitted.

A *financial market equilibrium* (FME) for a finance economy $\mathcal{E}$ is a tuple $\big((\bar{x}^i, \bar{z}^i)_{i=1,\ldots,I}, \bar{q}\big)$ with $\bar{q} \in Q$ such that:

1. $\bar{x}^i \in \arg\max\{u^i(x^i) | x^i \in B^i(\bar{q})\}$ for all $i = 1, \ldots, I$;

2. $V\bar{z}^i = \bar{x}^i_{\mathbf{1}} - \omega^i_{\mathbf{1}}$ for all $i = 1, \ldots, I$;

3. $\sum_{i=1}^{I} \bar{z}^i = 0$.

Note that the market-clearing conditions for the financial markets imply that the goods market also clears, since there is only one consumption good.

3 Existence of Equilibrium

Existence of equilibrium is proved using the space of asset prices $\bar{Q}$, defined by

$$\bar{Q} = \{q \in \mathbb{R}^{J+1} | q_0 \geq 0, \exists_{\pi \in \mathbb{R}^S_+} : q_{\mathbf{1}} = \pi V\}.$$

Clearly, the set $\bar{Q}$ is a finitely generated cone. Before proving a general existence theorem we present the following lemmata. The first lemma is standard and its proof is, therefore, omitted. Let $cl(Q)$ denote the closure of the set Q.

Lemma 2 *Under Assumption 3 it holds that* $\bar{Q} = cl(Q)$.

An important result needed to prove existence of an FME is the existence of a convergent sequence of state prices to the boundary. This lemma is crucial to our approach and makes it different from well-known proofs in the literature. Let ∂A denote the boundary of a set A in euclidean space.

Lemma 3 *Let* $(q^\nu)_{\nu \in \mathbf{N}}$ *be a sequence in* Q *converging to* $\bar{q} \in \partial\bar{Q}\backslash\{0\}$. *Then under Assumption 3 there exists a sequence of state prices* $(\pi^\nu)_{\nu \in \mathbf{N}}$ *in* $\mathbb{R}^S_{++}$ *satisfying* $q^\nu_{\mathbf{1}} = \pi^\nu V$ *for all* $\nu \in \mathbb{N}$ *and having a convergent subsequence with limit* $\bar{\pi} \in \mathbb{R}^S_+$ *satisfying* $\bar{q}_{\mathbf{1}} = \bar{\pi} V$. *Moreover, if* $\bar{q}_0 > 0$, *it holds that* $\bar{\pi} \in \partial\mathbb{R}^S_+$.

Proof. Define

$$\bar{Q}_{\mathbf{1}} = \{q_{\mathbf{1}} \in \mathbb{R}^J | \exists_{\pi \in \mathbb{R}^S_+} : q_{\mathbf{1}} = \pi V\}.$$

Since $\bar{Q}_{\mathbf{1}}$ is a finitely generated cone, it follows from Carathéodory's theorem (cf. Rockafellar (1970, Theorem 17.1)), that there exists a bounded sequence $(\hat{\pi}^\nu)_{\nu \in \mathbf{N}}$ in $\mathbb{R}^S_+$ such that $q^\nu_{\mathbf{1}} = \hat{\pi}^\nu V$ for every $\nu \in \mathbb{N}$.

For all $\nu \in \mathbb{N}$, since $q^\nu \in Q$, there exists $\tilde{\pi}^\nu \in \mathbb{R}^S_{++}$ such that $q^\nu_{\mathbf{1}} = \tilde{\pi}^\nu V$. Note that the sequence $(\tilde{\pi}^\nu)_{\nu \in \mathbf{N}}$ might not be bounded. Since $(\hat{\pi}^\nu)_{\nu \in \mathbf{N}}$ is bounded (in any given norm) by, say, $M > 0$, for all $\nu \in \mathbb{N}$, there exists a convex combination π^ν of $\tilde{\pi}^\nu$ and $\hat{\pi}^\nu$ that is bounded by $2M$, such that $\pi^\nu \in \mathbb{R}^S_{++}$. Clearly, for every $\nu \in \mathbb{N}$ it holds that $q^\nu_{\mathbf{1}} = \pi^\nu V$. Since $(\pi^\nu)_{\nu \in \mathbf{N}}$ is bounded there exists a convergent subsequence with limit, say, $\bar{\pi}$. Clearly, $\bar{q} = \bar{\pi} V$ and $\bar{\pi} \in \mathbb{R}^S_+$. Furthermore, when $\bar{q}_0 > 0$, it follows immediately from Assumption 3 that $\bar{\pi} \in \partial\mathbb{R}^S_+$. □

The following lemma concerns the boundary behaviour of the excess demand function.

Lemma 4 *Let* $(q^\nu)_{\nu \in \mathbf{N}}$ *be a sequence in* Q *with* $\lim_{\nu \to \infty} q^\nu = \bar{q} \in \partial\bar{Q}\backslash\{0\}$. *Under Assumptions 1–3 it holds that*

$$f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) \to \infty.$$

Proof. Suppose not. Then, for all $i = 1, \ldots, I$, since the sequence $(x^i_0(q^\nu) - \omega^i_0, Vz^i(q^\nu))_{\nu \in \mathbb{N}}$ is bounded from below, this sequence has a convergent subsequence with limit, say, $(\bar{f}^i_0, V\bar{f}^i_1)$. Without loss of generality we assume that the sequence itself converges to this vector. By Lemma 3 there exists a sequence $(\pi^\nu)_{\nu \in \mathbb{N}}$ in $\mathbb{R}^S_{++}$, satisfying $q^\nu_{\mathbf{1}} = \pi^\nu V$, for all $\nu \in \mathbb{N}$, and having a convergent subsequence with $\lim_{\nu \to \infty} \pi^\nu = \bar{\pi} \in \mathbb{R}^S_+$, satisfying $\bar{q}_{\mathbf{1}} = \bar{\pi} V$. Again, without loss of generality we assume that the sequence $(\pi^\nu)_{\nu \in \mathbb{N}}$ itself converges to $\bar{\pi}$.

Consider the case where $\bar{q}_0 > 0$ and $\bar{q}_{\mathbf{1}} \neq 0$. Let $\mathcal{S} = \{s | \bar{\pi}_s = 0\}$ and $\mathcal{S}^c = \{s | \bar{\pi}_s > 0\}$. Since $\bar{q}_{\mathbf{1}} \neq 0$ and since by Lemma 3, $\bar{\pi} \in \partial \mathbb{R}^S_+$, both sets are non-empty. Take $s^c \in \mathcal{S}^c$. Since $\omega_{s^c} > 0$, there exists an $i^c \in \{1, \ldots, I\}$ with $\omega^{i^c}_{s^c} > 0$. Let $\bar{x}^{i^c} = (\bar{f}^{i^c}_0 + \omega^{i^c}_0, V\bar{f}^{i^c}_{\mathbf{1}} + \omega^{i^c}_{\mathbf{1}})$. It holds that $\bar{q}_0 \bar{x}^{i^c}_0 + \bar{\pi} \bar{x}^{i^c}_{\mathbf{1}} = \bar{q}_0 \omega^{i^c}_0 + \bar{\pi} \omega^{i^c}_{\mathbf{1}}$ since u^{i^c} is continuous and strictly monotonic. Consider the bundle $\tilde{x}^{i^c} = \bar{x}^{i^c} + e(s)$ for some $s \in \mathcal{S}$, where $e(s) \in \mathbb{R}^{S+1}$ is the s-th unit vector. There exists $s^* \in \mathcal{S}^c$ satisfying $\bar{q}_0 \tilde{x}^{i^c}_0 + \bar{\pi}_{s^*} \tilde{x}^{i^c}_{s^*} > 0$ and thus $\tilde{x}^{i^c}_0 > 0$ or $\tilde{x}^{i^c}_{s^*} > 0$. Suppose first that $\tilde{x}^{i^c}_{s^*} > 0$. Since $u^{i^c}(\tilde{x}^{i^c}) > u^{i^c}(\bar{x}^{i^c})$ and u^{i^c} is continuous, it holds that

$$\exists_{\delta > 0} : u^{i^c}(\tilde{x}^{i^c} - \delta e(s^*)) > u^{i^c}(\bar{x}^{i^c}).$$

However, $\bar{q}_0 \tilde{x}^{i^c}_0 + \bar{\pi}(\tilde{x}^{i^c}_{\mathbf{1}} - \delta e_{\mathbf{1}}(s^*)) < \bar{q}_0 \omega^{i^c}_0 + \bar{\pi} \omega^{i^c}_{\mathbf{1}}$. Since $q^\nu_0 \to \bar{q}_0$ and $\pi^\nu \to \bar{\pi}$ we also have

$$\exists_{\nu_1 \in \mathbb{N}} \forall_{\nu > \nu_1} : q^\nu_0 \tilde{x}^{i^c}_0 + \pi^\nu (\tilde{x}^{i^c}_{\mathbf{1}} - \delta e_{\mathbf{1}}(s^*)) \leq q^\nu_0 \omega^{i^c}_0 + \pi^\nu \omega^{i^c}_{\mathbf{1}}.$$

Moreover, since $x^{i^c}(q^\nu) \to \bar{x}^{i^c}$ and u^{i^c} is continuous,

$$\exists_{\nu_2 \in \mathbb{N}} \forall_{\nu > \nu_2} : u^{i^c}(\tilde{x}^{i^c} - \delta e(s^*)) > u^{i^c}(x^{i^c}(q^\nu)).$$

So, for all $\nu > \max\{\nu_1, \nu_2\}$ we have $\tilde{x}^{i^c} - \delta e(s^*) \in B^{i^c}(q^\nu)$ and $u^{i^c}(\tilde{x}^{i^c} - \delta e(s^*)) > u^{i^c}(x^{i^c}(q^\nu))$, which contradicts $x^{i^c}(q^\nu)$ being a best element in $B^{i^c}(q^\nu)$. Suppose now that $\tilde{x}^{i^c}_0 > 0$. Using a similar reasoning as above we can show that there exists a $\delta > 0$ and $\nu^* \in \mathbb{N}$ such that for all $\nu > \nu^*$ it holds that $\tilde{x}^{i^c} - \delta e(0) \in B^{i^c}(q^\nu)$ and $u^{i^c}(\tilde{x}^{i^c} - \delta e(0)) > u^{i^c}(x^{i^c}(q^\nu))$, which contradicts $x^{i^c}(q^\nu)$ being a best element in $B^{i^c}(q^\nu)$. If $\bar{q}_0 = 0$ or $\bar{q}_{\mathbf{1}} = 0$ the proof follows along the same lines. □

Since $0 \in \partial \bar{Q}$ there is a tangent hyperplane at 0, i.e. there exists $\tilde{z} \in \mathbb{R}^{J+1} \backslash \{0\}$ such that $q\tilde{z} \geq 0$ for all $q \in \bar{Q}$. Since $\bar{Q}$ is full-dimensional, it holds that $q\tilde{z} > 0$ for all $q \in Q$. We show existence of FME by normalising asset prices to $q\tilde{z} = 1$, i.e. on a hyperplane parallel to the tangent hyperplane in 0. One possible choice for $\tilde{z}$ is the market portfolio z_M which is defined in the following way (cf. Herings and Kubler (2003)). Decompose the vector of total initial endowments in $\omega = \omega_M + \omega_\perp$, where $\omega_M \in \langle V \rangle$ and $\omega_\perp \in \langle V \rangle^\perp$, the null-space of $\langle V \rangle$. The market portfolio z_M is defined to be the unique portfolio satisfying $Vz = \omega_M$. If $\omega_M >> 0$ this implies $qz_M \geq 0$ for all no-arbitrage prices $q \in \bar{Q}$.

In the remainder, we fix $\tilde{z} \in \mathbb{R}^{J+1} \backslash \{0\}$ such that $q\tilde{z} > 0$ for all $q \in Q$. Denote the set of normalised prices by $\tilde{Q}$, i.e.

$$\tilde{Q} = \{q \in \bar{Q} | q\tilde{z} = 1\}.$$

Note that $\tilde{Q}$ can contain half-spaces and is, hence, not necessarily bounded. Based on the previous lemma, however, one can show that $f_0 + e^\top V f_{\mathbf{1}}$ becomes arbitrarily large by moving to the boundary of $\tilde{Q}$ or by taking $\|q\|_\infty$ large enough. Let $int\tilde{Q}$ denote the relative interior of $\tilde{Q}$.

Lemma 5 *Let $(q^\nu)_{\nu \in \mathbb{N}}$ be a sequence in $int\tilde{Q}$. Under Assumptions 1–3 it holds that*

1. $\lim_{\nu \to \infty} q^\nu = \tilde{q} \in \partial \tilde{Q} \Rightarrow f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) \to \infty$;
2. $\|q^\nu\|_\infty \to \infty \Rightarrow f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) \to \infty$.

Proof. In case 1 we have that $\tilde{q} \in \partial \bar{Q} \backslash \{0\}$, since $0 \notin \partial \tilde{Q}$. From Lemma 4 it follows that $f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) \to \infty$. In case 2, for all $\nu \in \mathbb{N}$ define $\tilde{q}^\nu = \frac{q^\nu}{\|q^\nu\|_\infty}$. Then $\tilde{q}^\nu \tilde{z} \to 0$. Moreover, for all $\nu \in \mathbb{N}$ it holds that $\|\tilde{q}^\nu\|_\infty = 1$. Hence, $(\tilde{q}^\nu)_{\nu \in \mathbb{N}}$ is bounded and therefore has a convergent subsequence with limit, say, $\tilde{q}$. Then $\tilde{q}\tilde{z} = \lim\limits_{\nu \to \infty} \tilde{q}^\nu \tilde{z} = 0$, i.e. $\tilde{q} \in \partial \bar{Q}$. Furthermore, $\|\tilde{q}\|_\infty = 1$ and hence $\tilde{q} \neq 0$. From Lemma 4 we know that $f_0(\tilde{q}^\nu) + e^\top V f_{\mathbf{1}}(\tilde{q}^\nu) \to \infty$. Since the budget correspondence is homogeneous of degree 0, we also get $f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) \to \infty$. □

With these lemmas in place, existence of an FME can be proved by using a direct approach as opposed to the indirect proof of e.g. Magill and Quinzii (1996).

Theorem 2 *Let $\mathcal{E}$ be a finance economy satisfying Assumptions 1–3. Then there exists an FME.*

Proof. A vector of prices $\bar{q} \in Q$ gives rise to an FME if and only if $f(\bar{q}) = 0$. For $\nu \in \mathbb{N}$, define the set Q^ν by

$$Q^\nu = \Big\{ q \in \tilde{Q} \Big| \frac{1}{\nu(\tilde{z}_0 + e^\top V \tilde{z}_{\mathbf{1}})} \leq q_0 \leq \frac{\nu}{\tilde{z}_0 + e^\top V \tilde{z}_{\mathbf{1}}},\ q_{\mathbf{1}} = \pi V, \\ \frac{1}{\nu(\tilde{z}_0 + e^\top V \tilde{z}_{\mathbf{1}})} \leq \pi_j \leq \frac{\nu}{\tilde{z}_0 + e^\top V \tilde{z}_{\mathbf{1}}} \text{ for all } j \Big\}.$$

Clearly, for every $\nu \in \mathbb{N}$, the set Q^ν is a non-empty convex and compact set in the relative interior of $\tilde{Q}$, and for every $q \in \tilde{Q}$ there exists $n \in \mathbb{N}$ such that $q \in Q^\nu$ for all $\nu \geq n$. Since the excess demand function f is continuous on $\tilde{Q}$ and therefore on every Q^ν, according to the stationary point theorem (cf. Eaves (1971) and Yang (1999)) there exists a stationary point of $f(\cdot)$ on every Q^ν, i.e.

$$\forall_{\nu \in \mathbb{N}} \exists_{q^\nu \in Q^\nu} \forall_{q \in Q^\nu} : q f(q^\nu) \leq q^\nu f(q^\nu) = 0.$$

Recall from Walras' law that $qf(q) = 0$ for all $q \in Q$. Consider the sequence $(q^\nu)_{\nu\in\mathbb{N}}$. If this sequence is unbounded or (has a subsequence that) converges to a point $\tilde{q} \in \partial\tilde{Q}$, then from Lemma 5 it follows that $f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) \to \infty$ and hence there exists $n \in \mathbb{N}$ such that $f_0(q^\nu) + e^\top V f_{\mathbf{1}}(q^\nu) > 0$ for all $\nu \geq n$. Since $(q_0, q_{\mathbf{1}}) = \frac{(1,e^\top V)}{\tilde{z}_0 + e^\top V \tilde{z}_{\mathbf{1}}} \in Q^\nu$ we obtain

$$q_0 f_0(q^n) + q_{\mathbf{1}} f_{\mathbf{1}}(q^n) = \frac{f_0(q^n) + e^\top V f_{\mathbf{1}}(q^n)}{\tilde{z}_0 + e^\top V \tilde{z}_{\mathbf{1}}} > 0,$$

which contradicts q^n being a stationary point of $f(\cdot)$ on Q^n. Hence, the sequence $(q^\nu)_{\nu\in\mathbb{N}}$ is bounded and (has a subsequence that) converges to a point $\bar{q} \in int\tilde{Q} \subset Q$. Moreover, $qf(\bar{q}) \leq 0$ for all $q \in int\tilde{Q}$ and so $\bar{q}$ solves the linear programming problem $\max\{qf(\bar{q}) | q\tilde{z} = 1\}$. From the dual of this problem it follows that $f(\bar{q}) = \lambda\tilde{z}$ for some $\lambda \in \mathbb{R}$. Using Walras' law we then obtain

$$0 = \bar{q}f(\bar{q}) = \lambda\bar{q}\tilde{z} = \lambda.$$

Hence, $\bar{q} \in Q$ and $f(\bar{q}) = \lambda\tilde{z} = 0$. □

4 Price Adjustments Towards Equilibrium

In this section we present a path of points in $\tilde{Q}$ from an arbitrary starting point in $\tilde{Q}$ to an FME. First, we prove the existence of such a path. Since $\tilde{Q}$ is a (possibly unbounded) polyhedron each point in $\tilde{Q}$ can be expressed as a convex combination of a finite number of points $\{v^1, \ldots, v^n\}$ plus a non-negative linear combination of directions $\{d^1, \ldots, d^m\}$. The recession cone of $\tilde{Q}$ is given by

$$re(\tilde{Q}) = \Big\{ q \in \mathbb{R}^{J+1} \Big| q = \sum_{k=1}^{m} \mu_k d^k, \mu \geq 0 \Big\}.$$

Let $q^0 \in \tilde{Q}$ be an arbitrary starting point in $\tilde{Q}$ and denote

$$\tilde{Q}_1 = conv(\{v^1, \ldots, v^n\}),$$

where $conv(A)$ denotes the convex hull of A. Then $\tilde{Q} = \tilde{Q}_1 + re(\tilde{Q})$. Note that $re(\tilde{Q})$ will be empty if $\tilde{Q}$ is bounded. We assume that the directions $\{d^1, \ldots, d^m\}$ are taken such that $q^0 + d^k \notin \tilde{Q}_1$, for all $k = 1, \ldots, m$. Define the polytope

$$\tilde{Q}(1) = \Big\{ q \in \tilde{Q} \Big| q = \sum_{h=1}^{n} \mu_h (v^h - q^0) + \sum_{k=1}^{m} \mu_{n+k} q^k + q^0, \mu_k \geq 0, \sum_{k=1}^{n+m} \mu_k \leq 1 \Big\},$$

and the set

$$K = \Big\{ q \in re(\tilde{Q}) \Big| q = \sum_{k=1}^{m} \mu_k q^k, \mu_k \geq 0, \sum_{k=1}^{m} \mu_k \leq 1 \Big\}.$$

Since $\tilde{Q} = \tilde{Q}_1 + re(\tilde{Q})$, the set $\tilde{Q}(1)$ is a J-dimensional polytope being a subset of $\tilde{Q}$. For simplicity we assume $q^0 \in int\tilde{Q}(1)$. We can now define the expanding set $\tilde{Q}(\lambda)$,

$$\tilde{Q}(\lambda) = \begin{cases} (1-\lambda)\{q^0\} + \lambda\tilde{Q}(1) & \text{if } 0 \le \lambda \le 1; \\ \tilde{Q}(1) + (\lambda - 1)K & \text{if } \lambda \ge 1. \end{cases}$$

Note that for all $\lambda \ge 0$ the set $\tilde{Q}(\lambda)$ is a polytope and that $\lim_{\lambda\to\infty} \tilde{Q}(\lambda) = \tilde{Q}$. In Figure 1 some of these sets are depicted.

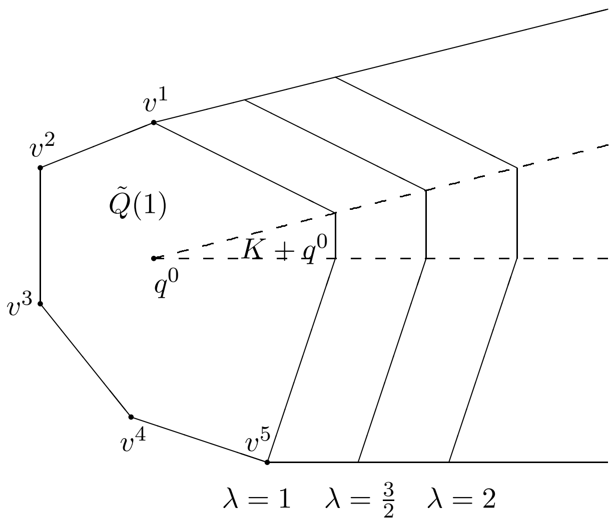

Figure 1: The expanding set $\tilde{Q}(\lambda)$.

By Lemma 5 we know that for all $M > 0$ there exists an $N > 0$ such that

$$\|q\|_\infty \ge N \Rightarrow f_0(q) + e^\top V f_{\mathbf{1}}(q) > M.$$

This implies that there exists a $\lambda^0 > 0$ such that for all stationary points $\bar{q}$ of f on $\tilde{Q}$ it holds that $\bar{q} \in \tilde{Q}(\lambda^0)$. Recall that all stationary points of f on $\tilde{Q}$ are FMEs. Let $\lambda^* = \max\{1, \lambda^0\}$ and define the homotopy $h : [0, \lambda^*] \times \tilde{Q}(\lambda^*) \to \tilde{Q}(\lambda^*)$ by

$$h(\lambda, q) = \begin{cases} proj_{\tilde{Q}(\lambda)}\big(q + f(q)\big) & \text{if } q \in \tilde{Q}(\lambda); \\ proj_{\tilde{Q}(\lambda)}\Big(proj_{\tilde{Q}(\lambda)}(q) + f(proj_{\tilde{Q}(\lambda)}(q))\Big) & \text{if } q \notin \tilde{Q}(\lambda), \end{cases}$$

where $proj_A(q)$ is the projection of q in $\|\cdot\|_2$ on the set A. Notice that the function h is continuous at every $(\lambda, q) \in [0, \lambda^*] \times \tilde{Q}(\lambda^*)$ because $\tilde{Q}(\cdot)$ is a continuous point-to-set mapping and every $\tilde{Q}(\lambda)$ is a compact and convex set. An important property of this homotopy is stated in the following lemma, where $(\lambda, q) \in [0, \lambda^*] \times \tilde{Q}(\lambda^*)$ is a fixed point of h if $h(\lambda, q) = q$.

Lemma 6 *Suppose that $(\lambda, q) \in [0, \lambda^*] \times \tilde{Q}(\lambda^*)$ is a fixed point of h. Then q is a stationary point of f on $\tilde{Q}(\lambda)$. If, in addition, $q \notin \partial\tilde{Q}(\lambda)$, then q is an FME.*

Proof. Let $(\lambda, q) \in [0, \lambda^*] \times \tilde{Q}(\lambda^*)$ be a fixed point of h. It is easy to see that $q \in \tilde{Q}(\lambda)$. This implies that

$$\begin{aligned} q = h(\lambda, q) &= proj_{\tilde{Q}(\lambda)}\big(q + f(Q)\big) \\ &= \arg\min_{q' \in \tilde{Q}(\lambda)} (q' - q - f(q))^\top (q' - q - f(q)). \end{aligned}$$

Hence, for all $q' \in \tilde{Q}(\lambda)$ we have that

$$\begin{aligned} &(q' - q - f(q))^\top (q' - q - f(q)) \geq f(q)^\top f(q), \\ \Longleftrightarrow\ &(q' - q)^\top (q' - q) \geq 2(q' - q)^\top f(q). \end{aligned}$$

Take $\hat{q} = \mu q' + (1-\mu)q$, for any μ, $0 < \mu \leq 1$. Since $\tilde{Q}(\lambda)$ is convex we have $\hat{q} \in \tilde{Q}(\lambda)$ and so for all $0 < \mu \leq 1$ it holds that

$$\begin{aligned} &(\hat{q} - q)^\top (\hat{q} - q) = \mu^2 (q' - q)^\top (q' - q) \geq 2\mu (q' - q)^\top f(q), \\ \Longleftrightarrow\ &\tfrac{1}{2}\mu (q' - q)^\top (q' - q) \geq (q' - q)^\top f(q). \end{aligned}$$

Letting $\mu \downarrow 0$, it follows that $q' f(q) \leq q f(q) = 0$, i.e. q is a stationary point of f on $\tilde{Q}(\lambda)$. Suppose $q \in \tilde{Q}(\lambda) \backslash \partial\tilde{Q}(\lambda)$. Then it holds that $h(\lambda, q) = q + f(q) \in \tilde{Q}(\lambda)$ and, hence, that $f(q) = 0$, since $h(\lambda, q) = q$. So, q is an FME. $\square$

From Lemma 6 and the proof of Theorem 2 it follows that for any fixed point (λ^*, q) of h, it holds that q is an FME. Combining Lemma 6 with Browder's fixed point theorem (see Browder (1960)) leads to the following result.

Theorem 3 *There exists a connected set C in $\tilde{Q}$ of stationary points of f connecting q^0 with an FME $\bar{q}$.*

Since the system of equations $h(\lambda, q) = q$ for $q \in \tilde{Q}$ has one degree of freedom, the connected set C is generically a path connecting the initial price vector q^0 with an equilibrium price vector. Such a path can be seen as being generated by a price adjustment process. In this process the prices are being adjusted in such a way that any price vector $\tilde{q} \in \partial\tilde{Q}(\lambda)$ being generated by the process maximises the value $q f(\tilde{q})$ of excess demand at $\tilde{q}$ over all $q \in \tilde{Q}(\lambda)$ until an equilibrium has been generated.

5 The Algorithm

There are simplicial algorithms to approximate the path of stationary points of an excess demand function f from an arbitrary starting point q^0 to an FME. The algorithm of Talman and Yamamoto (1989) finds a path to an equilibrium on a polytope. The algorithm should first be applied to $\tilde{Q}(1)$. If the algorithm terminates in $\bar{q} \in int\tilde{Q}(1)$, an approximating FME has been found. If it terminates at $q \in \partial\tilde{Q}(1)$, the algorithm is extended to the unbounded set $\tilde{Q} \setminus \tilde{Q}(1)$.

The algorithm generates a path of stationary points of a piecewise linear approximation to the excess demand function f. The set $\tilde{Q}(1)$ is a J-dimensional polytope and can be written as

$$\tilde{Q}(1) = \{q \in \mathbb{R}^{J+1} | q\tilde{z} = 1, a^i q \leq b_i, i = 1, \ldots, l\},$$

for some $a^i \in \mathbb{R}^{J+1} \backslash \{0\}$ and $b_i \in \mathbb{R}$, $i = 1, \ldots, l$.

Let $I \subset \{1, \ldots, l\}$. Then $F(I)$ is defined by

$$F(I) = \{q \in \tilde{Q}(1) | a^i q = b_i, i \in I\}.$$

The set $\mathcal{I} = \{I \subset \{1, \ldots, m\} | F(I) \neq \emptyset\}$ is the set of all index sets I for which $F(I)$ is a $(J - |I|)$-dimensional face of $\tilde{Q}(1)$. Let $q^0 \in int\tilde{Q}(1)$ be the starting point. For any $I \in \mathcal{I}$ define

$$vF(I) = conv(\{q^0\}, F(I)).$$

Now $\tilde{Q}$ is triangulated into simplices with finite mesh size in such a way that every $vF(I)$ is triangulated into $(J - |I| + 1)$-dimensional simplices.

Suppose that the algorithm is in $q^* \in vF(I)$, then q^* lies in some t-dimensional simplex $\sigma(q^1, \ldots, q^{t+1})$, its vertices being the affinely independent points $q^1, \ldots, q^{t+1}$, where $t = J - |I| + 1$ and $q^i \in vF(I)$ for all $i = 1, \ldots, t + 1$. There exist unique $\lambda_1^*, \ldots, \lambda_{t+1}^* \geq 0$, with $\sum_{i=1}^{\tau+1} \lambda_i^* = 1$, such that $q^* = \sum_{j=1}^{t+1} \lambda_j^* q^j$. The piecewise linear approximation of $f(\cdot)$ at q^* is then given by

$$\bar{f}(q^*) = \sum_{j=1}^{t+1} \lambda_j^* f(q^j).$$

Let λ, $0 < \lambda \leq 1$, be such that $q^* \in \partial\tilde{Q}(\lambda)$. Then $q^* = (1 - \lambda)q^0 + \lambda q'$, for some $q' \in F(I)$. For all $1 = 1, \ldots, m$, define $b_i(\lambda) = (1 - \lambda)a^i q^0 + \lambda b_i$. The point q^* is such that it is a stationary point of $\bar{f}$ on $\tilde{Q}(\lambda)$, i.e. q^* is a solution to the linear program

$$\max\{q\bar{f}(q^*) | a^i q \leq b_i(\lambda), i = 1, \ldots, m, q\tilde{z} = 1\}.$$

The dual problem is given by

$$\min\Big\{\sum_{i=1}^{m} \mu_i b_i(\lambda) + \beta \Big| \sum_{i=1}^{m} \mu_i a^i + \beta\tilde{z} = \bar{f}(q^*), \mu \geq 0, \beta \in \mathbb{R}\Big\}.$$

This gives a solution μ^*, β^*. Using the complementary slackness condition and assuming non-degeneracy we get the following:

$$\begin{aligned} I := & \{i | a^i q^* = b_i(\lambda)\} \\ = & \{i | a^i q' = b_i\} \\ = & \{i | \mu_i^* > 0\}. \end{aligned}$$

Hence,

$$\sum_{j=1}^{t+1} \lambda_j^* f(q^j) = \sum_{i \in I} \mu_i^* a^i + \beta^* \tilde{z},$$

$\sum_{j=1}^{t+1} \lambda_j^* = 1$, and $\mu_i^* \geq 0$, for all $i \in I$. In vector notation this system reads

$$\sum_{j=1}^{t+1} \lambda_j^* \begin{bmatrix} -f(q^j) \\ 1 \end{bmatrix} + \sum_{i \in I} \mu_i^* \begin{bmatrix} a^i \\ 0 \end{bmatrix} + \beta^* \begin{bmatrix} \tilde{z} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{3}$$

This linear system has $J + 2$ equations and $J + 3$ variables. The value β^* is a measure for how much the solution to the piecewise linear approximation deviates from Walras' law.

In each step of the algorithm one variable leaves and one new variable enters the basis of the linear system. This is achieved by making a linear programming pivot step in (3). Given that due to the pivot step a variable leaves the basis, the question is how to determine which variable enters the basis. First, suppose that some λ_k leaves the basis. This implies that q^* can be written as

$$q^* = \sum_{j=1, j \neq k}^{t+1} \lambda_j q^j.$$

Assuming non-degeneracy, q^* then lies in the interior of the facet τ of the simplex $\sigma(q^1, \ldots, q^{t+1})$ opposite to the vertex q^k. Now there are three possibilities. First, suppose that $\tau \in \partial v F(I)$ and $\tau \notin \partial \tilde{Q}(1)$. According to Lemma 4 this happens if and only if $\tau \subset v F(I \cup \{i\})$ for some $i \notin I$. Then we increase the dual dimension with one and μ_i enters the basis via a pivot step in (3). The second case comprises $\tau \subset \partial \tilde{Q}(1)$. Then the algorithm continues in $\tilde{Q} \backslash \tilde{Q}(1)$. The set τ is a facet of exactly one t-simplex σ' in the extension of $v(F(I))$ in $\tilde{Q}(2)$. The vertex opposite to τ of σ' is, say, q^k. The corresponding variable λ_k then enters the basis. Finally, it can be that $\tau \notin \partial v F(I)$. Then there is a unique simplex σ' in $v F(I)$ with vertex, say q^k, opposite to the facet τ. The corresponding variable λ_k then enters the basis.

The second possibility is that μ_i leaves the basis for some $i \in I$. Then the dual dimension is decreased with one, i.e. the set I becomes $I \backslash \{i\}$. Now there are two possibilities. If $I \backslash \{i\} = \emptyset$ then $\bar{f}(q^*) = \beta^* \tilde{z}$ and the algorithm terminates. The vector q^* is an approximate equilibrium and the algorithm can be restarted at q^* with a smaller mesh for the triangulation in order to improve the accuracy of the approximation. Otherwise, if $I \backslash \{i\} \neq \emptyset$, then define $I' = I \backslash \{i\}$. Since the primal dimension is increased with one there exists a unique simplex σ' in $vF(I')$ having σ as a facet. The vertex opposite to σ is, say, q^k. The algorithm continues with entering λ_k in the basis by means of a pivot step in (3). Assuming non-degeneracy in each step or taking lexicographic pivot steps the algorithm can not generate a simplex more than once. Since for any given $\lambda \geq 0$ the set $\tilde{Q}(\lambda)$ is covered by a finite number of simplices and for large enough λ the system (3) can have no solutions for any facet that lies in the boundary of $\tilde{Q}(\lambda)$, the algorithm must terminate within a finite number of steps with an approximating equilibrium.

The initial step of the algorithm consists of solving the linear program

$$\max\{qf(q^0) | a^i q \leq b_i,\ i = 1, \ldots, m,\ q\tilde{z} = 1\}.$$

Its dual program is given by

$$\min\Big\{\sum_{i=1}^{m} \mu_i b_i + \beta \Big| \sum_{i=1}^{m} \mu_i a^i + \beta\tilde{z} = f(q^0)\Big\}.$$

This gives as solution μ^0 and β^0. The set $F(I_0)$ is a vertex of $\tilde{Q}(1)$, where $I_0 = \{i \in \{1, \ldots, m\} | \mu_i^0 > 0\}$. There is a unique one-dimensional simplex $\sigma(w^1, w^2)$ in $vF(I_0)$ with vertices $w^1 = q^0$ and $w^2 \neq w^1$. Then λ_2 enters the basis by means of a pivot step in system (3).

As an example of this procedure consider the finance economy $\mathcal{E}(u, \omega, V)$ with two consumers, two assets and three states of nature. The utility functions are given by $u^1(x^1) = (x_0^1)^3 x_1^1 x_2^1 x_3^1$ for consumer 1 and $u^2(x^2) = x_0^2 x_1^2 x_2^2 (x_3^2)^2$ for consumer 2, and the initial endowments equal $\omega^1 = (1, 3, 3, 3)$ for consumer 1 and $\omega^2 = (4, 1, 1, 1)$ for consumer 2. On the financial markets, two assets are traded, namely a riskless bond and a contingent contract for state 3. That is,

$$V = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

It is easy to see that the set of no-arbitrage prices, Q, is given by

$$Q = \{(q_0, q_1, q_2) | q_0 > 0, q_2 > 0, q_1 > q_2\}.$$

Taking $\tilde{z} = (1, 1, 1)$, we get that

$$\tilde{Q} = \{q \in \mathbb{R}^3 | a^i q \leq 0, i = 1, 2, 3, q\tilde{z} = 1\},$$

where $a^1 = (-1, 0, 0)$, $a^2 = (0, 0, -1)$, and $a^3 = (0, -1, 1)$. Since $\tilde{Q}$ is a polytope one can set $\tilde{Q}(1) = \tilde{Q}$. The set $\tilde{Q}$ is the convex hull of the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 1/2, 1/2)$.

We start the algorithm at the price vector $q^0 = (\frac{5}{8}, \frac{1}{4}, \frac{1}{8})$. The grid size of the simplicial subdivision is taken to be $\frac{1}{8}$. In the first step of the algorithm we solve the linear program

$$\min\{\beta | \mu_1 a^1, \mu_2 a^2, \mu_3 a^3, \beta \tilde{z} = f(q^0), \mu_i \geq 0, i = 1, 2, 3\},$$

where $f(q^0) = (-3.02, 8.4667, -1.8333)$. This gives as solution $\mu^0 = (11.4867, 10.3, 0)$ and $\beta^0 = 8.4667$. This implies that $I^0 = \{1, 2\}$. The basic variables are λ_1, μ_1, μ_2, and β. The basis matrix corresponding to (3) equals

$$B = \begin{bmatrix} 3.02 & -1 & 0 & 1 \\ -8.4667 & 0 & 0 & 1 \\ 1.8333 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The first one-dimensional simplex that is generated is the simplex $\sigma(w^1, w^2) \in vF(I^0)$, where $w^1 = q^0$ and $w^2 = \frac{1}{64}(35, 22, 7)$. The algorithm proceeds by letting λ_2 enter the basis by means of a linear programming pivot step of $(-f(w^2), 1)$ into the matrix B^{-1}. This means, the algorithm leaves q^0 into the direction $vF(I^0) - q^0$ towards $vF(I^0) = (0, 1, 0)$. By doing so one finds that μ_2 leaves the basis. This implies that the dimension of the dual space is reduced and a two-dimensional simplex is generated in $vF(\{1\})$, namely $\sigma(w^1, w^2, w^3)$, where $w^3 = \frac{1}{64}(35, 20, 9)$. One proceeds by entering λ_3 into the basis by performing a pivot step in B^{-1}. In this way one obtains a sequence of two-dimensional adjacent simplices in $vF(\{1\})$ until the algorithm terminates when μ_1 leaves the basis. This happens after, in total, 12 iterations. The path of the algorithm is depicted in Figure 2.

The basic variables in the final simplex are λ_2, λ_3, λ_1, and β. The corresponding simplex is given by $\sigma(w^1, w^2, w^3)$, where $w^1 = \frac{1}{16}(5, 8, 3)$, $w^2 = \frac{1}{64}(15, 34, 15)$, and $w^3 = \frac{1}{64}(15, 36, 11)$. The corresponding values for λ are given by $\lambda_1 = 0.1223$, $\lambda_2 = 0.8460$, and $\lambda_3 = 0.0316$. This yields as an approximate FME the price vector

$$\bar{q} = \sum_{i=1}^{3} \lambda_i w^i = (0.2439, 0.5284, 0.2267).$$

The value of the excess demand function in $\bar{q}$ is given by $f(\bar{q}) = (-0.0174, 0.0145, -0.0151)$. The approximate equilibrium values for consumption at $t = 0$ and the demand for assets are given by $\bar{x}_0^1 = 3.7494$, $\bar{x}_0^2 = 1.2332$, $\bar{z}^1 = (-0.9794, -0.6756)$, and

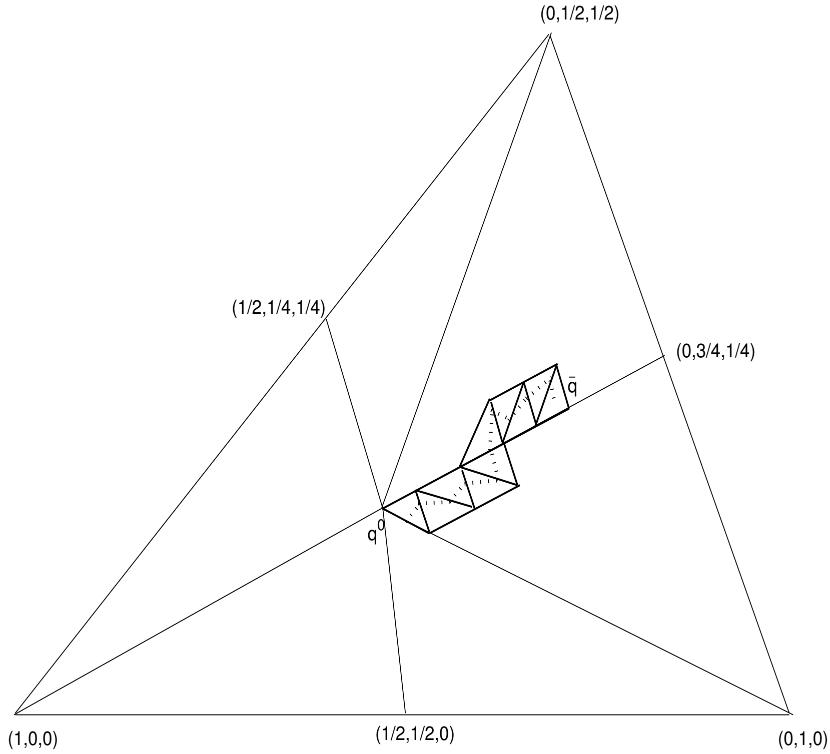

Figure 2: The path of prices (dotted line) generated by the algorithm.

$\bar{z}^2 = (0.9939, 0.6605)$, respectively. The accuracy of the approximation can be improved upon by restarting the algorithm in $\bar{q}$ and taking a smaller mesh size for the simplicial subdivision of $\tilde{Q}$.

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